

Model glasses coupled to two different heat baths

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Abstract. In a p -spin interaction spherical spin-glass model both the spins and the couplings are allowed to change with time. The spins are coupled to a heat bath with temperature T , while the coupling constants are coupled to a bath having temperature T_J . In an adiabatic limit (where relaxation time of the couplings is much larger than that of the spins) we construct a generalized two-temperature thermodynamics. It involves entropies of the spins and the coupling constants. The application for spin-glass systems leads to a standard replica theory with a non-vanishing number of replicas, $n = T/T_J$. For $p > 2$ there occur at low temperatures two different glassy phases, depending on the value of n . The obtained first-order transitions have positive latent heat, and positive discontinuity of the total entropy. This is an essentially non-equilibrium effect. The dynamical phase transition exists only for $n < 1$. For $p = 2$ correlation of the disorder (leading to a non-zero n) removes the known marginal stability of the spin glass phase. If the observation time is very large there occurs no finite-temperature spin glass phase. In this case there are analogies with the non-equilibrium (aging) dynamics. A generalized fluctuation-dissipation relation is derived.

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1 Introduction

The spin-glass model proposed by Edwards and Anderson [1,2] is a paradigm for large number of different random systems in nature. Its main assumptions are randomness, alternation and quenching of coupling constants, which reasonably reflect the crucial properties of random magnets with localized magnetic moments [2], structurally disordered materials [9], and large number of other systems. In particular, alternation means that there are ferromagnetic as well as antiferromagnetic couplings (attractive and repulsive in the language of particle dynamics in structurally disordered compounds and alloys [9]), and quenching means that there exists a large difference between relaxation times of couplings and magnetic degree of freedom (spins).

The typical example of a spin-glass is a dilution of a magnetic metal (such as Fe or Mn) in a non-magnetic host (for example Cu), where the concentration of the magnetic metal is not very large. Thus there is no direct exchange interaction between magnetic ions, but due to conduction electrons of Cu an indirect exchange interaction is possible (RKKY interaction) [9]. This interaction has oscillating character with respect of the distance between a pair of ions, and because the positions of magnetic ions in the

dilute are random we have the typical case of spin-glass. By a more simple and evident mechanism a spin-glass is realized in a dilution of ferromagnetic metal with antiferromagnetic one.

However, in all these cases there can be some doubts about truly quenching of coupling constants. Furthermore, several independent mechanisms can be pointed out for relatively slow changing of coupling constants with time: 1) diffusion of magnetic ions; 2) the distance between a pair of ions can be changed due to external variations of pressure [10]. Such reasons very naturally induce developments of spin-glass models where coupling constants are not quenched variables *a priori*, but change with time, according to some slow stochastic process. The particularly interesting case will arise if the characteristic mutual equilibration time between spins and coupling constants is very large, and there can be non-equilibrium states, where these subsystems are described by different temperatures T , T_J [24,28,29]. This is the general setup of the non-equilibrium thermodynamics [3,5–7]. As an example of this scenario the electron-ion plasma can be mentioned, where the mutual equilibration time is very large due to the large difference between masses of electron and ion [4]. To describe the situation, one can introduce two different thermostats with temperatures T and T_J . The conceptual

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and technical advantages of this approach were stressed by Bergmann and Lebowitz [6], and especially Meixner [7].

First phenomenological spin-glass models with time-dependent coupling constants were considered by Horner [22]. The two-temperature approach was introduced by Coolen *et al.* [24] and Dotsenko *et al.* [28]. On the other hand, a self-generated effective temperature appeared also in somewhat different context. It was employed to describe early glassy experiments [12,13], the long-time dynamics of certain models [14,15,17], and recently was put in the context of the non-equilibrium thermodynamics, analyzing exactly-solvable systems [16,37,41]. Our approach in the present paper is complementary: We assume the adiabatic separation of time-scales, and existence of two fixed temperatures for the different components.

We shall investigate the two-temperature dynamics and statics of the p -spin interaction spin glass model [32,35,37]. The statics of the model is described by one step of replica symmetry breaking (1RSB). The transition is first-order-type. Near the spin glass transition point the order parameter has a discontinuity [32,37], but there generally is no latent heat. The dynamics of the model is analytically tractable and has a rich structure [14,35]. Due to the systems sensitiveness to the interchange of the limits $N \rightarrow \infty$ and $t \rightarrow \infty$, there is a difference between the long time limit of the dynamics and the statics: The critical temperatures for the spin glass transition which are derived from the statics (where Gibbs distribution is assumed *a priori*) and from the dynamics (Langevin equations with typical initial conditions) are known to be different; this fact can be traced back to omission of activated processes in the mean-field dynamics [18,21,16]. There is a related difference in the thermodynamics of the system derived *via* the two approaches [38].

This paper is organized as follows. In Section 2 the dynamics of the model is introduced *via* a set of Langevin equations. In Section 3 we discuss the case where the coupling constants are fixed with respect of the dynamics of the spins (adiabatic statics). For this case we construct a generalized two-temperature thermodynamics. The considerations in this section are actually more general, and do not depend on the details of the adiabatic system. This general theory is applied to the concrete case of the mean-field spherical spin-glass model in Section 4. We also discuss different phase transitions which arise in this context. The detailed discussion of thermodynamical quantities is given.

In Section 5 the stationary (time-translation invariant) dynamic of the spin-glass model is investigated. We restrict ourselves to the high-temperature case, where this dynamics is physically relevant. In the adiabatic case it is shown that there is a dynamical phase transition point if the temperature of coupling constants is not smaller than the temperature of spins. Effects connected with large observation times are considered. The summary of our results is given at the last section. Some technical questions are considered in the appendices.

2 The multi-spin interaction model and its heat baths

In the mean-field spherical p -spin model we add a harmonic energy for the couplings, yielding

$$\mathcal{H} = \sum_{1 \leq i_1 < \dots < i_p \leq N} J_{i_1 \dots i_p} \sigma_{i_1} \dots \sigma_{i_p} + \frac{v}{2J_N^2} \sum_{1 \leq i_1 < \dots < i_p \leq N} J_{i_1 \dots i_p}^2, \quad (2.1)$$

where $J_N^2 = p!J^2/(2N^{p-1})$ is the usual normalization factor for mean-field models, with J and v being fixed energy scales. We assume that the coupling constants and the spins interact with heat baths at temperatures T_J and T , respectively. The (overdamped) Langevin equations for the dynamics in this model have the following form:

$$\Gamma \partial_t \sigma_i = -r \sigma_i - \frac{\partial \mathcal{H}}{\partial \sigma_i} + \eta_i(t),$$

$$\langle \eta_i(t) \eta_j(t') \rangle = 2\Gamma T \delta_{ij} \delta(t - t') \quad (2.2)$$

$$\bar{\Gamma}_J \partial_t J_{i_1 \dots i_p} = -\frac{\partial \mathcal{H}}{\partial J_{i_1 \dots i_p}} + \eta_{i_1 \dots i_p}(t),$$

$$\langle \eta_{i_1 \dots i_p}(t) \eta_{j_1 \dots j_p}(t') \rangle = 2\bar{\Gamma}_J T_J \delta_{i_1 \dots i_p, j_1 \dots j_p} \delta(t - t'). \quad (2.3)$$

Here $r(t)$ is the Lagrange multiplier for enforcing the spherical constraint $\sum_i \sigma_i^2(t) = N$, while Γ and $\bar{\Gamma}_J$ are the damping constants. The coupling constants $J_{i_1 \dots i_p}$ and the noises $\eta_{i_1 \dots i_p}$ are symmetric with respect of interchange of the indices. In equations (2.2, 2.3) so-called Einstein relation holds between the strength of noise and the damping constant. This means that the thermal baths themselves are in thermal equilibrium [8]. For ensuring the correct thermodynamical limit we must take: $\bar{\Gamma}_J = \Gamma_J/J_N^2$ (see Appendices A, B).

Straightforward calculations show that if $v \sim T_J$ and the limit $\Gamma_J \mapsto \infty$ is taken first, followed by $T_J \mapsto \infty$, then the coupling constants are quenched (with respect to the spins) independent Gaussian random variables. This limit thus yields the standard p -spin model (see Appendix A).

With help of standard methods [21,49] we study the dynamics in Appendix B. We arrive at the following equations for the average dynamics of a single spin in the mean field caused by the other ones

$$(\Gamma \partial_t + r)\sigma(t) = \frac{pJ^2}{2\Gamma_J} \int_{-\infty}^t d\bar{t} e^{-(t-\bar{t})/\tau_J} C^{p-1}(\bar{t}, t) \sigma(\bar{t}) + \frac{pT_J J^2}{2v} (p-1) \times \int_{-\infty}^t d\bar{t} e^{-(t-\bar{t})/\tau_J} C^{p-2}(\bar{t}, t) G(t, \bar{t}) \sigma(\bar{t}) + \eta(t),$$

$$\langle \eta(t) \eta(t') \rangle = 2\Gamma T \delta(t - t') + \frac{pT_J J^2}{2v} \exp(-|t - t'|/\tau_J) C^{p-1}(t, t'), \quad (2.4)$$

where

$$\tau_J = \frac{\Gamma_J}{v} \quad (2.5)$$

is the timescale at which the couplings change. Details of the derivation of this equation can be found in Appendix B.

Some comments about the general structure of equation (2.4) are at order. As it is well-known that the effective dynamical equations for spin-glass systems with quenched disorder are essentially non-Markovian, *i.e.*, they depend on the “history” of the process. Evidently, this arises due to quenching of a coupling constant with time. In our case — on account of the characteristic time (2.5) — the non-Markovian property is “smoothened” by the exponential kernel (see Eq. (2.4)).

With help of (2.4) we may derive coupled equations for the correlation function

$$C(t, t') = \frac{1}{N} \sum_i \langle \sigma_i(t) \sigma_i(t') \rangle \quad (2.6)$$

and the response function

$$G(t, t') = \frac{1}{N} \sum_i \frac{\delta \langle \sigma_i(t) \rangle}{\delta h_i(t')} \quad (2.7)$$

describing the response of spin σ_i to a small local field h_i imposed on an earlier moment t' , *via* an instantaneous change of the Hamiltonian as $\mathcal{H} \mapsto \mathcal{H} - h_i \sigma_i$. To fix the units, we shall take $\Gamma = 1$ from now on. We find for $t > t'$

$$\begin{aligned} (\partial_t + r)C(t, t') &= \frac{pJ^2}{2\Gamma_J} \int_{-\infty}^t d\bar{t} e^{-(t-\bar{t})/\tau_J} C^{p-1}(\bar{t}, t) C(\bar{t}, t') \\ &+ \frac{pT_J J^2}{2v} (p-1) \int_{-\infty}^t d\bar{t} e^{-(t-\bar{t})/\tau_J} C^{p-2}(t, \bar{t}) G(t, \bar{t}) C(\bar{t}, t') \\ &+ \int_{-\infty}^{t'} d\bar{t} G(t', \bar{t}) \left(2T\delta(t-\bar{t}) + \frac{pT_J J^2}{2v} e^{-|t-\bar{t}|/\tau_J} C^{p-1}(t, \bar{t}) \right) \end{aligned} \quad (2.8)$$

$$\begin{aligned} (\partial_t + r)G(t, t') &= \frac{pJ^2}{2\Gamma_J} \int_{-\infty}^t d\bar{t} e^{-(t-\bar{t})/\tau_J} C^{p-1}(\bar{t}, t) G(\bar{t}, t') \\ &+ \frac{pT_J J^2}{2v} (p-1) \int_{-\infty}^t d\bar{t} e^{-(t-\bar{t})/\tau_J} C^{p-2}(t, \bar{t}) G(t, \bar{t}) G(\bar{t}, t'). \end{aligned} \quad (2.9)$$

Generally speaking, both the relaxation toward a stationary state as well as fluctuations in that state are described by this closed pair of equations. In particular, in the second case the time-translational invariance is expected to hold: one-time quantities do not depend of time, two-time quantities depend only on the difference of times:

$$C(t, t') = C(t - t'), \quad G(t, t') = G(t - t'). \quad (2.10)$$

This regime only applies in the limit when the initial time t_0 goes to $-\infty$; this was already inserted in the lower limits

of integration of equation (2.9). Indeed, then the memory of the initial conditions is washed out, and the system relaxes toward its stationary state. This infinity is taken only after the thermodynamic limit ($N \rightarrow \infty$). $t - t'$ can be viewed as an observation time, or as the finding of some clock designed to display the temporal dynamics of fluctuations. It should be stressed that in its direct form, namely without additional physical mechanisms [34, 22, 35, 17], the stationary dynamics applies only for temperatures higher than the dynamical transition temperature T_d [46]. This is the case considered in the present paper (Sect. 5).

3 Two-temperature adiabatic thermodynamics: general structure

Recently Coolen, Penney, and Sherrington [24] proposed a dynamical approach to the statistical mechanics of spin glass systems, where the introduction of replicas is not needed initially (though they enter later without the $n \rightarrow 0$ limit; see also [23–28]). This approach can be called adiabatically static, because it is a static limit obtained by taking

$$\tau_J \mapsto \infty \quad (3.1)$$

immediately in the initial equations of motion (recall that τ_J remains much smaller than the initial time: $\tau_J \ll |t_0|$). For times much less than τ_J the spins will still see random couplings. As opposed to the standard case, the couplings are no longer uncorrelated. The correlation of couplings is coded in a finite T_J , and may lead to new physics.

Equations (2.2, 2.3) can be investigated by the method of adiabatic elimination (see for example [8]). Here we go further and construct the corresponding thermodynamics. For the case of a static distribution the procedure is as follows. First equation (2.2) is solved keeping the coupling constants J fixed (adiabatic following). Further, the Langevin equations in this case have the following equilibrium distribution

$$P(\sigma|J) = \frac{1}{Z_\sigma(J)} \exp[-\beta\mathcal{H}(\sigma, J)]. \quad (3.2)$$

(In this section we do not write the spherical factor explicitly; the reader can consider it to be included in $\mathcal{H}(\sigma, J)$.) The partition sum for given J -configuration is

$$Z_\sigma(J) = \text{Tr}_\sigma \exp[-\beta\mathcal{H}(\sigma, J)]. \quad (3.3)$$

In the evolution of the J -subsystem the averaging over the fast variables in (2.3) can be carried out. At quasi-equilibrium of the σ -subsystem this average can be performed and leads to the use of equation (3.2). In this way we get from equations (2.1, 2.2) and (2.3) a related dynamics for the couplings, in which $\mathcal{H}(\sigma, J)$ is replaced by $-T \ln Z_\sigma(J)$, which plays the role of effective Hamiltonian in the corresponding dynamics. Specifically one gets the effective equation of motion

$$\Gamma_J \partial_t J_{i_1 \dots i_p} = \partial_{J_{i_1 \dots i_p}} T \ln Z_\sigma(J) + \eta_{i_1 \dots i_p}(t). \quad (3.4)$$

As the noise is due to a bath at temperature T_J (see Eq. (2.3)), the equilibrium distribution of this equation reads

$$P(J) = \frac{Z_\sigma^n(J)}{\mathcal{Z}} \quad (3.5)$$

where

$$\mathcal{Z} = \int DJ Z_\sigma^n(J). \quad (3.6)$$

This approach introduces a “dynamical” replica index

$$n = \frac{T}{T_J}. \quad (3.7)$$

The free energy follows from the somewhat generalized formula

$$F = -\frac{T}{n} \ln \mathcal{Z} = -T_J \ln \mathcal{Z}. \quad (3.8)$$

Now we construct the appropriate thermodynamics for this two-temperature model using equations (3.5, 3.6). We first need the joint distribution of σ and J . One can express the unrestricted probability $P(\sigma, J)$ in terms of the conditional probability $P(\sigma|J)$ as

$$P(\sigma, J) = P(J)P(\sigma|J) = \frac{Z_\sigma^{n-1}}{\mathcal{Z}} \exp(-\beta\mathcal{H}(\sigma, J)). \quad (3.9)$$

The formula for the total energy U reads

$$U = \text{Tr}_\sigma \int DJ \mathcal{H}(\sigma, J)P(\sigma, J), \quad (3.10)$$

which can be viewed also as the average energy of the σ -subsystem. Direct calculations show that

$$-\frac{1}{n} \frac{\partial}{\partial \beta} \ln \mathcal{Z} \Big|_n = U, \quad -\frac{1}{n} \frac{\partial}{\partial \beta} \ln \mathcal{Z} \Big|_{T_J} = U - F_J, \quad (3.11)$$

where

$$F_J = - \int DJ T \ln Z_\sigma(J)P(J)$$

is the self-averaged free energy of the σ -subsystem or the mean effective energy.

The entropies of total system and its subsystems are defined by the usual Boltzmann-Gibbs-Shannon formula with help of the corresponding distributions (3.9). For example, the total entropy reads

$$S = - \int DJ \text{Tr}_\sigma P(\sigma, J) \ln P(\sigma, J). \quad (3.12)$$

This involves just the general, statistical definition of entropy for a macroscopic system, which is also relevant outside equilibrium [3]. Due to the decomposition (3.9) one gets two contributions,

$$S = S_\sigma + S_J \quad (3.13)$$

where

$$S_\sigma = \int DJ P(J) \{-\text{Tr}_\sigma P(\sigma|J) \ln P(\sigma|J)\} \quad (3.14)$$

is our analog of the usual quenched average entropy of the spin motion (more precisely, it is the so-called conditional entropy [3]), while the coupling-part of the entropy reads

$$S_J = - \int DJ P(J) \ln P(J). \quad (3.15)$$

The analogous separation of the total entropy in two or more parts appears in other problems of statistical physics [50,51]. In particular, it concerns the fine-graining procedure which introduces states of a statistical system relative to a fixed value of some properly chosen order parameter, which is a quantity under macroscopic control. Then the total entropy is also separated as in equation (3.13), where S_σ corresponds to average entropy of the relative states, and S_J corresponds to entropy of the order parameter itself. A simple example is just a piston separating a volume with a gas in two equivalent parts. Then S_σ is entropy of the gas in one part, and obviously $S_J = \ln 2$. A similar separation occurs in glassy transitions, where S_J corresponds to the configurational entropy or complexity [16,38]. The famous phenomena of Maxwell’s demon [50], and its subsequent reformulations display that in a process of measurement S_σ can be decreased, instead S_J increases, and the total entropy $S_\sigma + S_J$ can only increase.

Let us consider now a generalized thermodynamics which arises in the context of equations (3.8-3.15). In fact, to generalize the usual thermodynamics we notice the following relations only

$$F = F_J - T_J S_J \quad (3.16)$$

$$F = U - T_J S_J - T S_\sigma. \quad (3.17)$$

Equation (3.16) is the usual thermodynamical formula for the J -subsystem. The second formula is more interesting, since it connects the characteristics of the subsystems and the whole system. It agrees with the expression of the free energy for a glassy system put forward previously by one of us [16,39,40]. In that approach the equivalent of T_J is the dynamically generated effective temperature.

The results can be written in the differential form

$$dF = -S_\sigma dT - S_J dT_J. \quad (3.18)$$

If we add an external field, it can be checked that

$$dF = -S_\sigma dT - S_J dT_J - M dH. \quad (3.19)$$

This implies that the first law of thermodynamics takes the form

$$dU = T dS_\sigma + T_J dS_J - M dH. \quad (3.20)$$

As the last term can be identified with dW , the work done on the system, the change in heat reads

$$dQ = T dS_\sigma + T_J dS_J. \quad (3.21)$$

Equations (3.18, 3.19, 3.20) constitute a manifestation of a thermodynamic process in the following sense. A process is called “thermodynamic” if its characteristic time τ_{th} is much larger than internal relaxation times of the considered system. Due to this condition it is possible to represent the process as a chain of stationary states. In our adiabatic system there are two relaxation times, τ (an effective characteristic time of spins) and τ_J with $\tau_J \gg \tau$. There can thus exist two types of thermodynamic processes: A slow one with $\tau_{\text{th}} \gg \tau_J \gg \tau$ and a relatively fast one with $\tau_J \gg \tau_{\text{th}} \gg \tau$. In the second case the J -subsystem does not change during this process, implying, for example, $dS_J = 0$. We see that (3.18-3.20) represent a slow thermodynamic process where states of both subsystems are changed. This classification allows us to discuss the heating or cooling of our system. If the cooling is slow enough, typically both temperatures change; for example, in the extreme cooling process we can have $T_J \rightarrow 0$ and $T \rightarrow 0$ simultaneously. In the opposite case of a fast cooling (or heating) process T_J is a constant while T varies in time.

Irreversible effects also can be included in the present scheme. In general, irreversibility means that there are additional sources to increase entropy or decrease free energy. Namely it reads:

$$\begin{aligned} dF &< -S_\sigma dT - S_J dT_J - M dH, \\ dU &< T dS_\sigma + T_J dS_J - M dH. \end{aligned} \quad (3.22)$$

Equation (3.17) can be easily generalized to a many-level adiabatic system where the first part of variables is slow with respect to the second part, the second part is slow with respect to third part, ... For example, if we have a three-level system with the parts: $\{J\}$, $\{\sigma_1\}$, $\{\sigma_2\}$, having relaxation times $\tau_J \gg \tau_{\sigma_1} \gg \tau_{\sigma_2}$

$$F = U - T_J S_J - T_{\sigma_1} S_{\sigma_1} - T_{\sigma_2} S_{\sigma_2}. \quad (3.23)$$

It should be noticed, however, that on time scales of order τ_{σ_1} , where interesting non-equilibrium dynamics of the σ_1 system occurs, the σ_2 -system is in equilibrium, while the J -system is fixed; on the other hand, on timescales of order τ_J both the σ_1 and the σ_2 systems are in equilibrium. A physical realization of this scenario occurs in glass forming liquids, with their fast and slow β processes, while the J -system then describes the configurational or α -processes. In this context equation (3.23) corresponds also to recently introduced models with two-level disorder: Coupling constants are also considered as frozen variables with respect to some other set of variables [25,26].

We should stress the main difference of the presented generalized thermodynamics compared to the usual, well-known nonequilibrium thermodynamical description [3,5]. The stationary distribution (3.2, 3.5) can be arbitrary far from equilibrium. It is close to the corresponding Gibbsian distribution in the limit of a small difference between T and T_J , or for weakly-interacting subsystems σ and J .

4 Adiabatic statics

In this section we investigate the adiabatic static limit of the mean field spherical spin-glass model introduced in the Section 2. The free energy is described by (3.8). As in reference [32] we have

$$\mathcal{Z} = \prod_{\alpha\beta} \left(\int \frac{dq_{\alpha\beta} d\lambda_{\alpha\beta}}{2\pi i} \right) \exp(-NG_n(q_{\alpha\beta}, \lambda_{\alpha\beta})) \quad (4.1)$$

$$\begin{aligned} 2G_n(q_{\alpha\beta}, \lambda_{\alpha\beta}) &= -n \ln 2\pi - \frac{\mu}{p} \sum_{\alpha\beta} q_{\alpha\beta}^p + \sum_{\alpha\beta} q_{\alpha\beta} \lambda_{\alpha\beta} \\ &\quad + \ln \det(-\lambda) \\ &= \text{const.} - \frac{\mu}{p} \sum_{\alpha\beta} q_{\alpha\beta}^p - \text{tr} \ln(q), \end{aligned} \quad (4.2)$$

where $\mu = pT_J J^2 / 2vT^2$, $q_{\alpha\beta} = \langle \sigma_\alpha \sigma_\beta \rangle$ is the usual order parameter describing the spin-glass ordering, and $\lambda_{\alpha\beta}$ are Lagrange multipliers [1,2]¹. We have three independent parameters: T , $T_J = T/n$, and v . In this paper we consider phase transitions only in the following subspace of three dimensional space of the parameters: $v = T_J$, $n = T/T_J$ is fixed, and the relevant parameter is T (of course, all such regimes with $v \sim T_J$ are qualitatively equivalent). Thus for μ we have the standard expression

$$\mu = \frac{p\beta^2 J^2}{2}. \quad (4.3)$$

As we have discussed in the previous section, there is some regime of cooling where n indeed can be a constant.

4.1 Replica symmetric solution

In the investigation of a spin-glass the first step is to make the Replica Symmetry (RS) assumption for the order parameter [1,2]: $q_{\alpha\beta} = q$ (for $\alpha \neq \beta$), where q is the usual Edwards-Anderson parameter. In other words, one assumes that there is only one thermodynamical state (up to possible global symmetry transformations). The expression for the RS free energy $f_{\text{rs}} = F_{\text{rs}}/N$ has the following form

$$\begin{aligned} 2\beta f_{\text{rs}} &= -\ln(1-q) - \frac{1}{n} \ln \left(1 + \frac{nq}{1-q} \right) \\ &\quad - \frac{\mu}{p} (1 + (n-1)q^p) \end{aligned} \quad (4.4)$$

where q is determined by the saddle point equation

$$\mu q^{p-1} = \frac{q}{(1-q)(1+q(n-1))}. \quad (4.5)$$

¹ Besides (4.2) there is a contribution to the free energy which arises from the integration by the coupling constants, and has an order $\mathcal{O}(N^p \ln N)$. Usually this contribution is omitted (see [28] for example), because it does not depend on the order parameter. An alternative point of view is to consider (4.2) as the leading finite-size effect.

In Section 5 we shall see that this equation can be obtained from the long-time statics if a slow dynamics for the coupling constants is assumed. Following (3.12-3.17) we get for energy and entropies

$$2\beta u_{rs} = -\frac{\mu}{p}(1 + (n-1)q^p) \quad (4.6)$$

$$2s_\sigma = \ln(1-q) + \frac{q}{1-q+nq} - \frac{\mu}{p}(1-q^p) \quad (4.7)$$

$$2s_J = \ln\left(1 + \frac{nq}{1-q}\right) - \frac{nq}{1-q+nq} + \frac{n\mu}{p}(1-q^p). \quad (4.8)$$

It is well-known that from a point of view of phase transitions the set of p -spin models can be divided into the two main groups: $p > 2$ and $p = 2$. In the first case the qualitative phase diagram of the model does not depend on p (as long as it is finite). Thus in the main part of this subsection we investigate the case $p > 2$; our results for $p = 2$ will be presented at the end.

We start with investigation of equation (4.5). For high temperatures there is only the paramagnetic phase with $q = 0$. The critical point $T_{1,rs}$ can be defined as the first temperature where a non-zero solution of (4.5) is possible. For $p > 2$ there occurs a first-order phase transition point with discontinuity of the order parameter

$$q_{1,rs} = \frac{\sqrt{(2-n)^2(p-1)^2 + 4p(p-2)(n-1)} + (n-2)(p-1)}{2p(n-1)}. \quad (4.9)$$

The related onset temperature is

$$T_{1,rs} = J \sqrt{\frac{p}{2} q_{1,rs}^{p-2} (1 - q_{1,rs})(1 + (n-1)q_{1,rs})}. \quad (4.10)$$

Some limiting cases can be investigated; for example, if p is large enough we get

$$q_{1,rs} \sim 1 - \frac{1}{p}, \quad T_{1,rs} \sim \sqrt{\frac{n}{2e}}. \quad (4.11)$$

Any physical solution must be stable against small perturbations, therefore the analysis of linear stability for a possible replica-symmetric solutions should be performed. The eigenvalues of the corresponding Hessian for a finite n were computed in [32]. There are three main sectors of fluctuations and the three corresponding eigenvalues:

$$\Lambda_1 = -\mu(p-1)q^{p-2} + \frac{1}{(1-q)^2}, \quad \sum_{\beta} \delta q_{\alpha\beta} = 0 \quad (4.12)$$

$$\Lambda_2 = \Lambda_1 - \frac{(n-2)q}{(1-q)^2(1+(n-1)q)}, \quad \sum_{\beta} \delta q_{\alpha\beta} \neq 0, \quad \sum_{\alpha\beta} \delta q_{\alpha\beta} = 0 \quad (4.13)$$

$$\Lambda_3 = \Lambda_1 - \frac{(n-1)q}{(1-q)^2(1+(n-1)q)} \left(2 - \frac{nq}{1+(n-1)q} \right), \quad \sum_{\alpha\beta} \delta q_{\alpha\beta} \neq 0. \quad (4.14)$$

The first eigenvalue, the so-called ‘‘replicon’’ or ‘‘ergodon’’, is displayed by the most coherent fluctuations (RSB is checked usually by this eigenvalue); the third eigenvalue corresponds to the most non-coherent fluctuations, and the second one takes an intermediate position. To investigate the stability of the non-zero RS solution, we first discuss which eigenvalue is relevant for different values of n , so which is the smallest one. A simple analysis shows that for $n < 1$ Λ_1 is relevant, while for $n > 1$ Λ_3 is the most dangerous one. This result is important: the relevant sector of the fluctuations depends on n , implying that the whole structure of the phase space has strong dependence on n also. At $n = 1$ Λ_2 becomes relevant too, but it causes no extra problem, since we have $\Lambda_1 = \Lambda_2$.

For $p > 2$ the paramagnetic solution $q = 0$ is stable everywhere (the case $p = 2$ will be discussed at the end of this section). Now we check the stability of the nonzero solution of equation (4.5) for $n < 1$. This solution monotonically decreases from (4.9) to zero. From positivity of Λ_1 we get

$$q \geq \frac{p-2}{n+p-2}. \quad (4.15)$$

Thus the nonzero solution is stable only for

$$T \leq T_{rs,st} = J \sqrt{\frac{n^2 p (p-1) (p-2)^{p-2}}{2(n+p-2)^p}}. \quad (4.16)$$

Because $T_{rs,st} < T_{1,rs}$ we see that the solution is stable only for sufficiently low temperatures. In particular, for $n = 0$ it is unstable for every temperature, and that explains why this solution could be discarded till now. On the other hand, we shall show that if $n \geq 1$ the RS solution is stable everywhere. Indeed, for the nonzero solution of (4.5) Λ_3 is positive, and has the following form:

$$\Lambda_3 = \frac{p(n-1)q^2 + (2-n)(p-1)q + 2-p}{(1-q)^2(1+(n-1)q)^2}. \quad (4.17)$$

Further discussion of properties of replica symmetric spin-glass solution will be given after consideration of replica symmetry breaking solutions. In particular, we shall see that there are other phase transitions.

4.1.1 The case $p = 2$

Let us now consider the special case $p = 2$. It is well-known that the model with $n = 0$ is described statically by a replica symmetric ansatz [30] and has non-trivial dynamical properties [30, 52]. For example, the Λ_1 eigenvalue vanishes everywhere in the low temperature spin-glass phase.

For our case A_1 is always positive for nonzero n , since it holds that

$$A_1 = \frac{nq}{(1-q)^2(1+(n-1)q)} > 0. \quad (4.18)$$

So the correlation between the couplings removes the zero modes and stabilizes the structure of the vacuum. Since $A_3(p=2, n>1)$ is also a positive function we see that for $p=2$ the RS solution is stable for all n , and for all T . In other words, the statics of the model with $p=2$ is completely described by the RS ansatz. In particular, one gets for $p=2$

$$q_{1,rs} = \frac{(|n-2| + n - 2)}{4(n-1)}. \quad (4.19)$$

Now if $n \leq 2$ the phase transition is second-order. For $n > 2$ the phase transition is first-order with the following scenario: besides $T_{1,rs}$, where the spin-glass phase occurs at the first time, there is also the transition point $T_{2,rs} < T_{1,rs}$, that is determined by comparing free energies of paramagnet and spin-glass. These statements can be illustrated analytically in the case of positive but small $n-2$. Starting from equation (4.4) we get:

$$q_{1,rs} = \frac{n-2}{2}, \quad q_{2,rs} = \frac{2(n-2)}{3}, \quad (4.20)$$

$$T_{1,rs} = J \left(1 + \frac{(n-2)^2}{8} \right), \quad T_{2,rs} = J \left(1 + \frac{(n-2)^2}{9} \right). \quad (4.21)$$

The difference between free energies of the spin-glass phase and paramagnet is positive at the transition point $T = T_{1,rs}$:

$$\Delta f(T_{1,rs}) = J \frac{(n-2)^4}{384} > 0. \quad (4.22)$$

It means that the spin-glass phase appears as metastable one for $n > 2$. The free energies are equal at the second transition point: $\Delta f(T_{2,rs}) = 0$. Therefore this point must be considered as the temperature of the phase transition from paramagnet to spin-glass. Indeed, for $T < T_{2,rs}$ we have $\Delta f(T) < 0$: The spin-glass phase has lower free energy compared to that of the paramagnet.

The final phase diagram for $p=2$ is depicted in Figure 6.

We have seen that a finite n removes the marginal states for the $p=2$ model. In the Section 5 we shall see that a similar statement holds also for the case $p > 2$. But then the condition $n > 1$ is needed for stabilizing the corresponding marginal states.

4.2 Replica symmetry breaking

Now we investigate Replica Symmetry Breaking (RSB) solutions. As we have seen in the previous subsection, for $n > 1$ the stability of a RS spin-glass solution gives some

Table 1. $T_{1,rs}$, $T_{2,rs}$ and $q_{1,rs}$ for different n ($n > 1$), and $p = 3$. At $T = T_{1,rs}$ the RS spin-glass phase first appears as metastable one. The thermodynamical first-order phase transition from the paramagnetic phase occurs at $T = T_{2,rs}$ ($J = 1$).

n	$T_{1,rs}$	$T_{2,rs}$	$q_{1,rs}$
1.01	0.6125	0.5862	0.5012
1.10	0.6271	0.6001	0.5118
2.50	0.8270	0.7543	0.5954
4.00	1.0052	0.9213	0.6228
6.00	1.2049	1.1140	0.6378
10.0	1.5289	1.3567	0.6496

hint about irrelevance of RSB in this range of n . We shall, however, first discuss the case $n < 1$. Only the first step of RSB (1RSB) is considered, because a more general type of RSB is not possible in this model; we omit the proof, since it can be found in [32].

Taking the usual steps [1,2] we get the following equations

$$\begin{aligned} 2\beta f_{rsb} = & -\frac{m-1}{m} \ln(1-q_1) \\ & -\frac{n-m}{nm} \ln(1-(1-m)q_1 - mq_0) \\ & -\frac{1}{n} \ln(1-(1-m)q_1 + (n-m)q_0) \\ & -\frac{\mu}{p}(1-q_1^p + m(q_1^p - q_0^p) + nq_0^p) \end{aligned} \quad (4.23)$$

where $q_{\alpha\beta}$ takes the values q_1 and q_0 , and m is the RSB parameter. For $n=0$ we recover the usual 1RSB equations [32]. Following (3.12-3.17) we get for energy and entropies

$$2\beta u_{rsb} = -\frac{\mu}{p}(1-q_1^p + m(q_1^p - q_0^p) + nq_0^p) \quad (4.24)$$

$$\begin{aligned} 2s_\sigma = & \frac{m-1}{m} \ln(1-q_1) + \frac{1}{m} \ln(1-(1-m)q_1 - mq_0) \\ & + \frac{q_0}{1-(1-m)q_1 + (n-m)q_0} \\ & -\frac{\mu}{p}(1-q_1^p + m(q_1^p - q_0^p)) \end{aligned} \quad (4.25)$$

$$\begin{aligned} 2s_J = & -\ln(1-(1-m)q_1 - mq_0) + \ln(1-(1-m)q_1) \\ & + (n-m)q_0 - \frac{nq_0}{1-(1-m)q_1 + (n-m)q_0} \\ & + \frac{n\mu}{p}(1-q_1^p + m(q_1^p - q_0^p)). \end{aligned} \quad (4.26)$$

The saddle point equations are

$$\mu(q_1^{p-1} - q_0^{p-1}) = \frac{q_1 - q_0}{(1-q_1)(1-(1-m)q_1 - mq_0)} \quad (4.27)$$

$$\begin{aligned} \mu q_0^{p-1} = & \\ & \frac{q_0}{(1-(1-m)q_1 - mq_0)(1-(1-m)q_1 - (m-n)q_0)}. \end{aligned} \quad (4.28)$$

The parameter m is determined by its saddle point equation,

$$\begin{aligned} \frac{\mu}{p}(q_1^p - q_0^p) &= \frac{1}{m^2} \ln \left(1 + \frac{m(q_1 - q_0)}{1 - q_1} \right) \\ &- \frac{n - m}{nm} \frac{q_1 - q_0}{(1 - (1 - m)q_1 - mq_0)} \\ &- \frac{1}{n} \frac{q_1 - q_0}{(1 - (1 - m)q_1 - (m - n)q_0)}. \end{aligned} \quad (4.29)$$

There is another possibility to fix m : As was shown before [21, 35, 37], this parameter can be fixed also by the so called ‘‘marginality condition’’. The resulting theory describes metastable states, in a manner also monitored by the dynamics [35] (see also Sect. 5). In this paper we consider only the purely static condition (4.29).

First we note that there is a remarkable analogy between the RS finite- n free energy (4.4) and the free energy of the 1RSB solution with $q_0 = 0$. Interchanging q_1 with q and m with n correspondingly, we arrive at the identical expressions. This analogy between corresponding free energies was considered in the SK model with similar but more complicated techniques including two types of frozen variables [25]. Recently XY spin-glass model has been investigated by the same approach [26]. We shall show that a similar, perhaps more informative analogy can be found in the dynamics of the present model.

As known, the physical interpretation of replica symmetry breaking is connected with decomposition of the phase space into pure states (ergodic components) [1, 2]. This structure is contained in the overlap $q_{\alpha\beta}$. In particular, for 1RSB the values q_1, q_0 can be interpreted as the self-overlap of a pure state and the mutual overlap between two pure states. This information is coded in the probability distribution of overlaps:

$$\begin{aligned} P(q) &= \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} \delta(q - q_{\alpha\beta}) \\ &= \frac{1-m}{1-n} \delta(q - q_1) + \frac{m-n}{1-n} \delta(q - q_0). \end{aligned} \quad (4.30)$$

It is seen that for the interpretation of $(1-m)/(1-n)$ and $(m-n)/(1-n)$ as probabilities we need:

$$\begin{aligned} n < m < 1 &\text{ for } n < 1 \\ 1 < m < n &\text{ for } n > 1. \end{aligned} \quad (4.31)$$

One could expect that conditions (4.31) are satisfied automatically, if other more obvious physical conditions (for example, $q_1 > q_0$) are valid. However, it is not so. Later we shall show that they should be considered as additional conditions selecting the correct solution.

4.2.1 Replica symmetry breaking with vanishing lower plateau

Let us now discuss the solution of equations (4.27, 4.29) for the case $q_0 = 0$. Then the solution itself becomes independent on n . However, the dependence on n does enter

through equations (4.30, 4.31). The considered solution has partly been investigated in [32, 33]. First we note that there is a convenient parametrisation of equations (4.27, 4.29) [32, 33]. If we denote

$$c = \frac{mq_1}{1 - q_1} \quad (4.32)$$

then for this quantity we get the temperature independent equation

$$\frac{c^2}{p} = (1 + c) \ln(1 + c) - c. \quad (4.33)$$

The positive solution of this equation should be selected. (The authors of [32] employ the slightly different variable $y = 1/(c + 1)$.) Taking this into account, the equation for q_1 reads

$$\mu q_1^{p-1} = \frac{q_1}{(1 + c)(1 - q_1)^2}. \quad (4.34)$$

The highest temperature for which this equation has a non-zero solution will be denoted by $T_{1,\text{rsb}}$:

$$T_{1,\text{rsb}} = J \sqrt{\frac{2(1+c)}{p} \left(\frac{p-2}{p} \right)^{p-2}}, \quad (4.35)$$

and q_1 at this point has the value

$$q_1(T_{1,\text{rsb}}) = \frac{p-2}{p}. \quad (4.36)$$

Further, using (4.32) we get

$$m(T_{1,\text{rsb}}) = \frac{2c}{p-2}. \quad (4.37)$$

This value is greater than 1 for all $p > 2$. On the other hand, the parameter m monotonically decreases with temperature from (4.37) at $T = T_{1,\text{rsb}}$ to zero at $T = 0$ (see (4.43)). It means that for $n > 1$ only a part of the solution from $m(T_{1,\text{rsb}})$ to $m = 1$ can be physically permissible; otherwise we get physically meaningless results for the weights (4.30). Namely, if n is in the interval $1 < n < m(T_{1,\text{rsb}})$, then the temperature, where the solution appears as physical one, will be determined from the condition $m(T) = n$. In the opposite case, where $m(T_{1,\text{rsb}}) < n$ that temperature is just $T_{1,\text{rsb}}$ itself.

On the other hand, in case $n < 1$ the physical part is consistent only with $n < m < 1$. Namely, for $n < 1$ the possible transition point must be determined from the condition $m = 1$. The transition to the RSB spin-glass phase with vanishing lower plateau, and $m = 1$ at the critical point, has been found in [32] for $n = 0$. It occurs at the temperature

$$T_{2,\text{rsb}} = J \sqrt{\frac{p}{2(1+c)} \left(\frac{1+c}{c} \right)^{2-p}}. \quad (4.38)$$

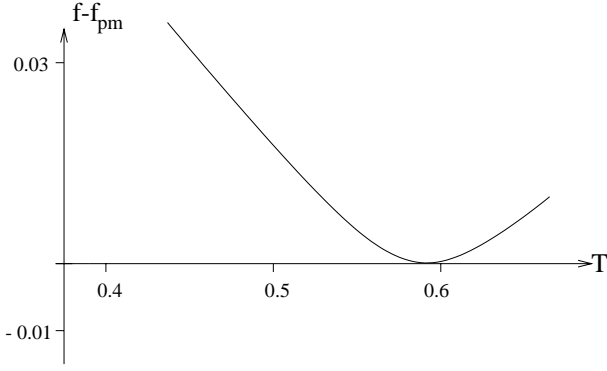


Fig. 1. The free energy (subtracted the paramagnetic contribution) of the 1RSB spin-glass phase with vanishing lower plateau *vs.* temperature ($p = 3$). For $n > 1$ ($n < 1$) a part of the right-hand (left-hand) branch of the presented curve should be chosen as the physically permissible one (see Eq. (4.30)). It turns out that the chosen part of the right-hand branch always corresponds to a metastable phase.

At this point q_1 jumps from zero to

$$q_1(T_{2,\text{rsb}}) = \frac{c}{c+1}, \quad (4.39)$$

and it goes monotonously to unity when T tends to zero. The transition is intermediate between first-order and second-order: the order parameter has a jump but the energy and entropy are continuous, as we see from equations (4.24–4.26).

Note that the free energy in the spin-glass phase is higher than that of the paramagnetic state (Fig. 1). It is usual for this type of phase transitions [1, 2, 44, 32].

If $p \mapsto \infty$ then also $c \mapsto \infty$:

$$\frac{J^2}{2T_{2,\text{rsb}}^2} \sim (\ln c - 1) \exp\left(\frac{1}{\ln c - 1}\right) \mapsto \infty \quad (4.40)$$

$$q_1 \sim 1 - \frac{1}{c}. \quad (4.41)$$

This behavior in the large- p limit is in the sharp contrast with the case of p -interaction Ising spin-glass [31], where phase transition point is finite when $p \mapsto \infty$.

Let us now consider the zero temperature behavior of the solution with fixed but not very large p . A simple analysis shows that in this case

$$1 - q_1 \sim \frac{T}{J} \sqrt{\frac{2}{p(1+c)}}, \quad (4.42)$$

$$m \sim \frac{T}{J} c \sqrt{\frac{2}{p(1+c)}}, \quad (4.43)$$

$$f(T \mapsto 0) = -\frac{J(c+p)}{\sqrt{2p(1+c)}}. \quad (4.44)$$

Note again that the free energy of the solution remains finite in the zero-temperature limit in contrast to the paramagnetic free energy which tends to minus infinity.

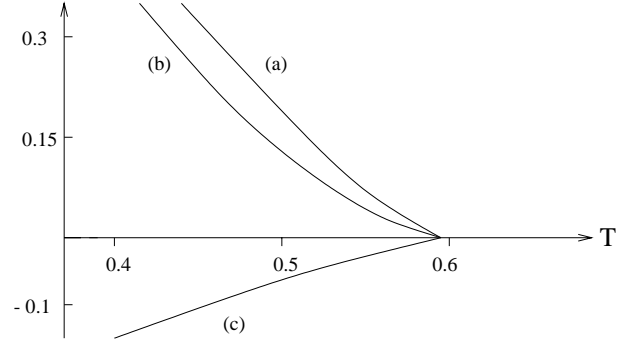


Fig. 2. The entropies (subtracted the corresponding paramagnetic contributions; see Eqs. (4.25, 4.26)) of the 1RSB spin-glass phase with vanishing lower plateau *vs.* temperature ($p = 3$, $n = 0.5$). (a) s_σ , entropy of the spins; (b) s , the total entropy; (c) s_J , entropy of the coupling constants. The curves start at the transition point $T = T_{2,\text{rsb}}$.

The considered solution is stable and all relevant eigenvalues of the Hessian are nonzero at the phase transition point. In particular, for the relevant eigenvalue: (see [32] for the derivation)

$$\Lambda_1 = -\mu(p-1)q_1^{p-2} + \frac{1}{(1-q_1)^2}, \quad (4.45)$$

we get from equation (4.34):

$$\Lambda_1 = \frac{1}{(1-q_1)^2} \left(1 - \frac{p-1}{1+c}\right) > 0. \quad (4.46)$$

The eigenvalue (4.45) describes fluctuations deep inside a pure state. As usual, critical slowing down is absent at this static first order phase transition. Later we shall show that in the long-time dynamics there is a dynamical phase transition for all $n < 1$.

It is of interest to discuss the behavior of entropies s_σ , s_J , and s in the vicinity of the transition point $T_{2,\text{rsb}}$. In particular, we have from equations (4.24–4.26)

$$2s_\sigma = \ln(1-q_1) + \frac{1}{m} \ln(1+c) + \frac{\mu}{p} (1 - (1-m)q_1^p) \quad (4.47)$$

$$2s_J = \frac{n\mu}{p} (1 - (1-m)q_1^p). \quad (4.48)$$

In spite of the jump of q_1 , they change continuously from the paramagnetic phase to the spin-glass one. We shall compare their behavior in spin-glass and paramagnet at the same value of temperature. For $T < T_{2,\text{rsb}}$ the entropy of the spins becomes higher in the spin-glass phase: $s_\sigma > s_\sigma(q_1 = 0)$. On the other hand, the entropy of the coupling constants is lower: $s_J < s_J(q_1 = 0)$, as it is obvious from equation (4.48). Consequently, the total entropy $s = s_\sigma + s_J$ is also higher in the spin-glass phase as compared to its value for $q_1 = 0$. The behavior of the entropies is presented in Figure 2. Further properties of the RSB spin-glass phase (in particular, for $n > 1$) will be discussed later when constructing the complete phase diagram.

4.2.2 Replica symmetry breaking with non-vanishing lower plateau

We made an extensive but unsuccessful search for additional RSB phases. The 1RSB solution with the vanishing lower plateau is the only physically relevant one. Here we present one of those attempts, which allows us to illustrate the importance of condition (4.31). A solution with q_0 emerging smoothly from $q_0 = 0$ is not compatible with the saddle-point equations (4.27). Therefore we investigate a continuous transition from the RS solution with $q \equiv q_1 > 0$ and a RSB solution with $q_1 > 0$, $q_0 > 0$ but small $\Delta q = q_1 - q_0$. With this assumption we have the following equations at the transition point

$$\mu_c(p-1)q_c^{p-2} = \frac{1}{(1-q_c)^2} \quad (4.49)$$

$$\mu_c q_c^{p-2} = \frac{1}{(1-q_c)(1+(n-1)q_c)}. \quad (4.50)$$

The first of these expresses that the replicon eigenvalue Λ_1 of a RS solution with $q = q_1$ vanishes. The transition point is thus given by

$$T = Jn \sqrt{\frac{p(p-2)^{p-2}(p-1)}{2(p-2+n)^p}}. \quad (4.51)$$

The order parameters $q_0 = q_1$ have the following jump at the phase transition point

$$q_c = \frac{p-2}{p-2+n}. \quad (4.52)$$

As we see, it is the same point where stability of the RS spin-glass solution is restored. There is a useful parametrization of equations (4.27–4.29):

$$x = \frac{q_0}{q_1}, \quad c = \frac{m\Delta q}{1-q_1}, \quad (4.53)$$

then we have

$$\frac{1-x^p - px^{p-1}(1-x)}{p(1-x^{p-1})(1-x)} = \frac{(1+c)\ln(1+c) - c}{c^2}. \quad (4.54)$$

Using equations (4.53, 4.54) we get at the transition point:

$$m_c = \frac{n}{2}. \quad (4.55)$$

It can be proved that m decreases with decreasing of temperature, and $m = 0$ when $T = 0$. According to our discussion at equations (4.30, 4.31) it means that a 1RSB phase with non-vanishing lower plateau cannot be considered as a physical one, at least not for $n < 2$. For $n > 2$ and then $m(T) > 1$ this is unphysical, since the condition $q_1 > q_0$ is violated (this condition does not depend on n , and it is necessary for the interpretation of q_0 and q_1 as overlaps).

This behavior is in sharp contrast with SK model [29, 28] where the main effect of finite n is to introduce a non-vanishing lower plateau. This plateau increases

with n , and replica symmetry breaking disappears at some critical value. It should be noted that in our case only the property (4.31) forbids the existence of the considered phase for $n < 2$. All other requirements are satisfied: it is stable, and has well-defined free energy. Stability can be checked by positivity of the following eigenvalues

$$\Lambda_1 = -\mu(p-1)q_1^{p-2} + \frac{1}{(1-q_1)^2}, \quad (4.56)$$

$$\Lambda_0 = -\mu(p-1)q_0^{p-2} + \left(\frac{1-q_1+nq_0}{(1-q_1)(1-q_1+m\Delta q+nq_0)} \right)^2. \quad (4.57)$$

We should mention the possibility of more general RSB solutions in our model. For $n = 0$ it was proven [32] that 1RSB solutions are the most general RSB ones, and more orders of RSB are impossible. This proof can be generalized also for $n > 0$. The physical meaning of this statement is that a finite n introduces correlations between coupling constants partially removing frustrations, and cannot lead to more complicated phase space with more orders of RSB.

4.3 The static phase diagram

In this subsection we construct the phase diagram of our model by considering all physically relevant solutions (it means stable and with the correct probability of overlaps (4.30)): paramagnet, RS spin-glass, RSB spin-glass with vanishing lower plateau.

4.3.1 The case $n < 1$

First we discuss the case $n < 1$. For high temperatures the system is in the paramagnetic phase. If temperature decreases, then at $T = T_{2,\text{rsb}}$ (see Eq. (4.38)) the RSB spin-glass with vanishing lower plateau appears. Its free energy is greater than the paramagnetic one (see Fig. 1) but, nevertheless, it is chosen as the relevant phase. It is the usual choice, and a possible argument is a hypothesis about a non-perturbative instability of the paramagnetic state below $T_{2,\text{rsb}}$. As far as we know, there is no convincing proof of this statement. There is only some hint gained from an analysis of finite-size corrections in a Potts glass [44]. As we have mentioned already, this phase transition is first-order with respect to the order parameter, but second-order with respect to derivatives of free energy. In particular, the latent heat (the difference between the energies of the high temperature and the low temperature phase at the transition point) vanishes.

At the present stage we shall go back to the RS spin-glass solution, and analyze its free energy. Recall that this phase is stable for $T < T_{\text{rs,st}}$ (see Eq. (4.16)), and for some range of n we have $T_{\text{rs,st}} > T_{2,\text{rsb}}$ (see Fig. 3). In particular, the sign of

$$\begin{aligned} \Delta f_{\text{rs}} &\equiv f_{\text{rs}} - f_{\text{para}} \\ &= -\frac{T}{2} \ln(1-q) - \frac{T}{2n} \ln \left(1 + \frac{nq}{1-q} \right) + \frac{\beta J^2}{4} (1-n)q^p \end{aligned} \quad (4.58)$$

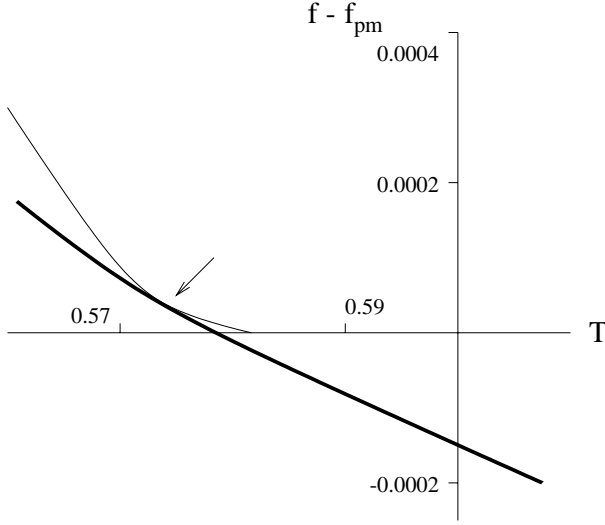


Fig. 3. Free energies (subtracted the paramagnetic free energy) *vs.* temperature ($n = 0.955$, $p = 3$). Thick line: the RS spin-glass phase; normal line: the RSB spin-glass phase with vanishing lower plateau. The arrow denotes the phase transition point occurring at $T = T_{3,rs}$ from the 1RSB spin-glass phase to the RS one.

should be checked. In this respect the interval $0 < n < 1$ is divided into two subintervals. For $n_0 < n < 1$ (where n_0 is some positive value, to be discussed later) it holds that $\Delta f_{rs}(T_{rs,st}) < 0$. Taking into account that at $T = T_{rs,st}$ RS spin-glass first appears as a physical solution, we conclude that in this range of temperatures it should be considered as metastable with respect to the paramagnet, in spite of its lower free energy. The opposite point of view will mean that $T = T_{rs,st}$ must be considered as the point of a phase transition which is meaningless, because $\Delta f_{rs}(T_{rs,st}) \neq 0$. If temperature is decreasing further we get the point $T = T_{2,rs}$ where $\Delta f_{rs}(T_{2,rs}) = 0$, and $\Delta f_{rs}(T_{rs,st}) > 0$ for $T_{2,rs} > T$. This behavior of the free energy is presented by Figures 3, 4. However, this point is always lower than the transition temperature from the paramagnet into the RSB spin-glass phase with vanishing lower plateau: $T_{2,rsb} > T_{2,rs}$. (On the other hand $T_{2,rsb} \leq T_{2,rs}$ for $1 \leq n$, but this case is physically different and will be discussed a bit later.) Therefore, the temperature $T = T_{2,rs}$ is not considered as the thermodynamical phase transition point towards the most stable phase. Certainly, it is only the point where RS spin-glass phase becomes more stable than the paramagnetic state.

For $n < n_0$ one has $\Delta f_{rs} > 0$, so the RS spin-glass state at the beginning appears with higher free energy than the paramagnet. n_0 is defined by the condition $\Delta f_{rs}(T_{rs,st}) = 0$. In Table 2 we represent the values of n_0 for different p . In particular, we see that $n_0(p)$ decreases with increasing p .

According to equations (4.30, 4.31) for $n < 1$ any 1RSB spin glass cannot exist as a physical one if $m < n$. From Section 4.2.1 we know that for the RSB spin-glass with vanishing lower plateau, m monotonically decreases with temperature from $m = 1$ at $T = T_{2,rsb}$ to $m = 0$ at $T = 0$. It means that a phase transition should exist from a RSB

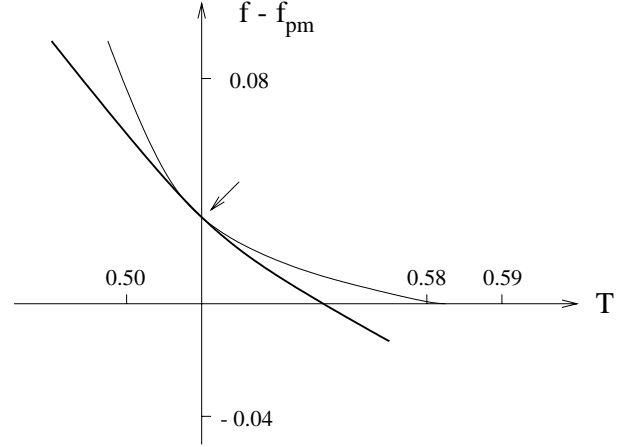


Fig. 4. Free energies (subtracted the paramagnetic contribution) *vs.* temperature ($n = 0.8$, $p = 3$). Thick line: the RS spin-glass phase; normal line: the RSB spin-glass phase with vanishing lower plateau. The arrow denotes the phase transition point at $T = T_{3,rs}$.

Table 2. n_0 for different p ($J = 1$).

n_0	p
0.68632	3
0.61605	4
0.53108	6
0.36803	18
0.31670	30

spin-glass with vanishing lower plateau to a RS spin-glass at a temperature $T_{3,rs}$ defined by

$$m(T_{3,rs}) = n. \quad (4.59)$$

As equations (4.23–4.29) show, the order parameters are equal at this point: $q_1 = q$. It is also easy to show that the free energies, energies and entropies s_σ , s_J of the corresponding phases are also equal. Thus, at $T = T_{3,rs}$ the order parameter q_0 jumps from zero to q_1 , ensuring the replica-symmetric behavior for $T < T_{3,rs}$. On the other hand, q_1 changes continuously. Both the RS spin-glass and the RSB spin-glass with vanishing lower plateau are stable at and around $T = T_{3,rs}$. In this sense the transition at $T = T_{3,rs}$ is very similar to the transition occurring from the paramagnetic state at $T = T_{2,rsb}$, where q_0 changes continuously (*i.e.*, remains zero) but q_1 has a jump. It is worth to note that for $T < T_{3,rs}$ the free energy of the RS spin-glass is lower than the free energy (more exactly its analytical continuation) of the RSB spin-glass with vanishing lower plateau. Different values of $T_{3,rs}$ are presented in Table 3.

In summary, we conclude that when temperature is decreasing for fixed $n < 1$ there are only two temperatures of thermodynamical phase transitions: 1) $T = T_{2,rsb}$ is the transition point from the paramagnet to the RSB spin-glass with vanishing lower plateau. 2) $T = T_{3,rs}$ is the transition point from the RSB spin-glass into the RS phase. These transitions are quite similar by their physical

Table 3. The transition temperature $T_{3,rs}$ from the 1RSB spin-glass phase to the RS one, for different n ($n < 1$), and $p = 3$ ($J = 1$).

n	$T_{3,rs}$
0.90	0.5575
0.80	0.5240
0.70	0.4805
0.60	0.4413
0.25	0.2326

meaning. The main distinction is in the difference between free energies of the corresponding low-temperature and the high-temperature phase. The temperatures $T_{rs,st}$ and $T_{2,rs}$ also have certain physical meanings, but are not considered as true transition temperatures.

4.3.2 The case $n > 1$

In the case $n > 1$ there are three relevant solutions: the RS spin-glass ($q > 0$), the paramagnet ($q = 0$), and the RSB spin-glass with vanishing lower plateau in the region $1 < m < n$. Recall that this last state formally exist for $T < T_{1,rsb}$ (see Eq. (4.35)), but the region of its actual existence must be chosen according to the value of n as it has been done for $n < 1$. Doing so we immediately conclude that for $n > 1$ the RSB spin-glass with vanishing lower plateau cannot be considered as the most stable one, because its free energy is always higher than the free energy of the paramagnetic state (Fig. 1). It can be viewed only as a metastable one in the above-mentioned region. The behavior of the corresponding free energy is presented in Figure 1. Therefore, our attention will be restricted only to the paramagnet and the RS spin-glass phase.

According to our discussion in Section 4.1, the RS spin-glass phase first appears at $T = T_{1,rs}$ (Eq. (4.9)) but its free energy at this point is higher than the paramagnetic one. These free energies are equal when $T = T_{2,rs}$, and for $T < T_{2,rs}$ the RS spin-glass phase has lower free energy. Thus, the temperature $T_{2,rs}$ is the thermodynamical phase transition point. This transition is first order, and it is connected with a jump of the order parameter q . It is obvious from equation (4.6) that for $T < T_{2,rs}$ the mean energy of the spin-glass phase is also lower than the paramagnetic one. It means that the latent heat is positive as for (usual) first-order phase transitions in equilibrium systems. Values of $T_{1,rs}$, and $T_{2,rs}$ are represented in Table 1. In particular, this scenario of phase transition is the usual one for multi-spin interaction ferromagnets [9,20], for some metamagnetic materials [9] or for phase transitions in a compressible lattice [11]. For such a system the jump of entropy at the transition point is negative (*i.e.*, a low-temperature phase has lower entropy) according to the usual relation $F = E - TS$ between free energy, energy and entropy. At this stage it should be recalled again that our system is not in the usual equilibrium, and we are considering phase transitions in the nonequilibrium steady state. In particular, we have the basic relation (3.17) be-

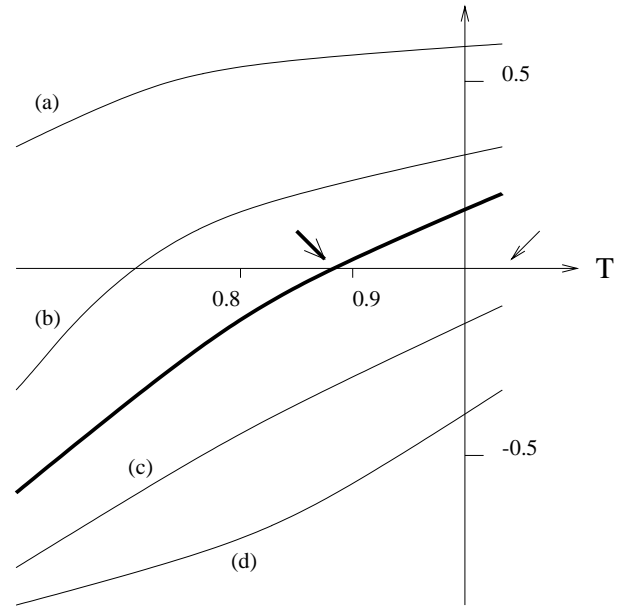


Fig. 5. Thermodynamical functions (minus the corresponding paramagnetic contributions) of the RS spin-glass ($n = 4$, $p = 3$) vs. temperature. The thick arrow denotes the phase transition point from paramagnet to the RS spin-glass phase at $T = T_{2,rs}$. The normal arrow denotes the temperature $T = T_{1,rs}$, where the RS spin-glass solution appears first as a metastable phase. Thick line: free energy; (a) s_J , entropy of the coupling constants; (b) s , the total entropy; (c) u_{rs} , mean energy; (d) s_σ , entropy of the spins.

tween free energy, energy, and entropies. At the transition point $T = T_{2,rs}$ this relation can be written as

$$\Delta u = T \Delta s_\sigma + T_J \Delta s_J, \quad (4.60)$$

where $\Delta u = u_{rs} - u_{pm}$, $\Delta s_\sigma = s_{\sigma,rs} - s_{\sigma,pm}$, and $\Delta s_J = s_{J,rs} - s_{J,pm}$ are differences between the corresponding quantities of the RS spin-glass and the paramagnet, and u_{rs} , $s_{\sigma,rs}$, $s_{J,rs}$ are defined by equations (4.6–4.8). Further, T and T_J are connected through $T = T_{2,rs}$. In particular, positive latent heat means $\Delta u < 0$, and consequently $T \Delta s_\sigma + T_J \Delta s_J < 0$. However, it is interesting to know the signs of Δs_σ , Δs_J , and $\Delta s = \Delta s_\sigma + \Delta s_J$ separately because these quantities have independent physical meanings, extensively discussed in Section 3. We get

$$\Delta s > 0, \Delta s_J > 0, \Delta s_\sigma < 0. \quad (4.61)$$

The behavior of Δs is in the sharp contrast with usual, equilibrium first-order transitions. This somewhat surprising fact should deserve further attention. The behavior of various thermodynamical quantities near $T_{2,rs}$ is presented in Figure 5. The difference between spins entropies Δs_σ becomes positive starting from some temperature lower than $T_{2,rs}$. In other words, for sufficiently low temperatures both entropies are higher in the spin-glass phase. Let us recall in this context that free energy and mean energy of the spin-glass phase are lower than the paramagnetic ones for sufficiently low temperatures.

The positivity of the latent heat, and the result (4.61) holds also for the first-order phase transitions occurring

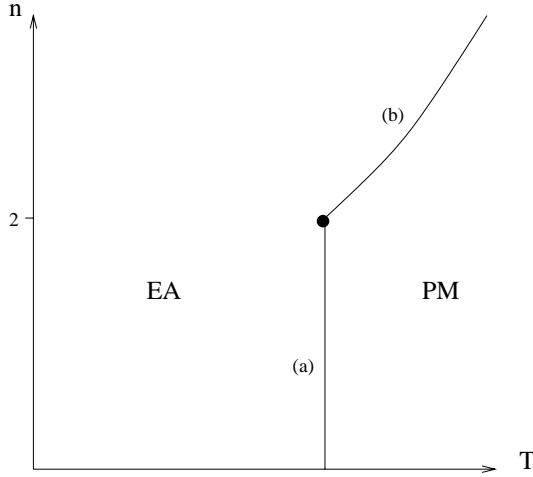


Fig. 6. The static phase diagram of the model in the n - T plane for $p = 2$. The paramagnetic phase is denoted by PM, and EA means the RS or Edwards-Anderson spin-glass phase. (a) $T = J$, the line of the second-order transitions; (b) $T = T_{2,rs}(n)$, the line of the first-order transitions with a positive jump of the total entropy. The thick dot indicates the multicritical point.

for $p = 2$, $n > 2$ (see Sect. 4.1.1). In the case of positive but small $n - 2$ we can get analytical expressions for Δs_σ , Δs_J , and Δs using equations (4.20–4.22). The jump $q_{2,rs}$ of the order parameter is positive but small at $T = T_{2,rs}$, and we get to leading order

$$\begin{aligned}\Delta s_\sigma &= -\frac{1}{2}q_{2,rs}^2 + \mathcal{O}(q_{2,rs}^3) < 0, \\ \Delta s_J &= \frac{1}{2}q_{2,rs}^2 + \mathcal{O}(q_{2,rs}^3) > 0, \\ \Delta s &= \frac{5}{24}q_{2,rs}^3 + \mathcal{O}(q_{2,rs}^4) > 0.\end{aligned}\quad (4.62)$$

This is an analytical illustration of the more general result (4.61).

The special attention should be devoted to the case $n = 1$. The free energy, and the energy of the RS spin-glass coincide with their analogs for the paramagnetic phase. However, from the point of view of the order parameter, the phase transition occurs to the RS spin-glass phase (when decreasing temperature for a fixed $n = 1$). Further, q increases monotonically from $q = (p - 2)/(p - 2 + n)$ at $T = T_{2,rsb} = T_{2,rs} = T_{3,rs}$ to $q = 1$ at $T = 0$. The phase transition is first-order for $p > 2$, and second-order for $p = 2$. It is interesting that s_σ and s_J depend on q even for $n = 1$, however, they compensate each other so that their sum $s = 0$. The final phase diagram is presented in Figure 7.

5 Stationary dynamics

In this section the stationary (time-translation invariant) dynamics of our model will be investigated. In particular, we study the dynamical phase transition, compare the predictions of the long-time dynamics with the static ones, as

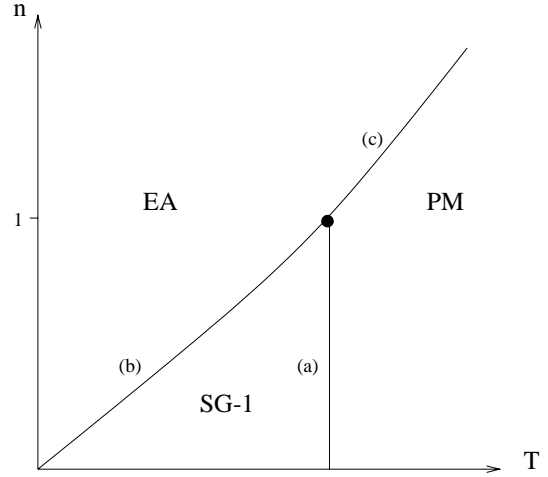


Fig. 7. The static phase diagram of the model in the n - T plane for $p > 2$. The paramagnetic phase is denoted by PM, and SG-1, EA mean correspondingly the RSB spin-glass phase with vanishing lower plateau, and the RS spin-glass phase. (a) $T = T_{2,rsb}(p)$; (b) $T = T_{3,rs}(n, p)$; (c) $T = T_{2,rs}(n, p)$. The lines (a) and (b) indicate first-order transitions without latent heat; (c) indicates the first-order transition with the positive latent heat, and a positive jump of the total entropy. The thick dot indicates the multicritical point.

well as investigate effects connected with very large observation times. The case when temperature is lower than the dynamical transition point will not be attached, because the assumption of stationarity does not hold there without additional mechanisms (regularization) [22,35]. equations (2.8, 2.9) have the following form in the time-translation invariant regime

$$\begin{aligned}(\partial_t + r)C(t) &= \frac{pJ^2}{2\Gamma_J} \int_0^\infty d\bar{t} e^{-\bar{t}/\tau_J} C^{p-1}(\bar{t})C(t-\bar{t}) \\ &+ \frac{pT_J J^2}{2v} (p-1) \int_0^\infty d\bar{t} e^{-\bar{t}/\tau_J} C^{p-2}(\bar{t})G(\bar{t})C(t-\bar{t}) \\ &+ \frac{pT_J J^2}{2v} \int_0^\infty d\bar{t} e^{-(t+\bar{t})/\tau_J} C^{p-1}(t+\bar{t})G(\bar{t}),\end{aligned}\quad (5.1)$$

$$\begin{aligned}(\partial_t + r)G(t) &= \frac{pJ^2}{2\Gamma_J} \int_0^t d\bar{t} e^{-(t-\bar{t})/\tau_J} C^{p-1}(t-\bar{t})G(\bar{t}) \\ &+ \frac{pT_J J^2}{2v} (p-1) \int_0^t d\bar{t} e^{-(t-\bar{t})/\tau_J} C^{p-2}(t-\bar{t})G(t-\bar{t})G(\bar{t}).\end{aligned}\quad (5.2)$$

5.1 Adiabatic dynamics: not very large observation times

Not very large observation times are defined by the condition $t \ll \tau_J$: The spins relax to their stationary state, while the coupling constants are fixed. The usual form of FDT

$$\partial_t C(t) = T(-G(t) + G(-t)) \quad (5.3)$$

does hold in this case, since when the coupling constants are frozen the violation of the detailed balance condition (which arises in account of a difference between temperatures, and is responsible for a possible breaking of equilibrium relations) cannot be effective [43]. Applying FDT one gets from equation (5.1)

$$\begin{aligned} (\partial_t + r)C(t) &= \frac{pJ^2}{2\Gamma_J} \int_0^\infty d\bar{t} e^{-\bar{t}/\tau_J} C^{p-1}(\bar{t})C(t-\bar{t}) \\ &+ \frac{pT_J J^2}{2v} (p-1) \int_0^\infty d\bar{t} e^{-\bar{t}/\tau_J} C^{p-2}(\bar{t})C(t-\bar{t})G(\bar{t}) \\ &+ \frac{pT_J J^2}{2v} \int_0^\infty d\bar{t} e^{-\bar{t}/\tau_J} G(\bar{t})C^{p-1}(t+\bar{t}). \end{aligned} \quad (5.4)$$

In the last two integrals one can take $\exp(-\bar{t}/\tau_J) \sim 1$, since $G(\bar{t})$ already restricts the relevant domain of integration to $\bar{t} \ll \tau_J$. In the first integral we find after a change of the variable:

$$\frac{1}{\Gamma_J} \int_0^\infty d\bar{t} e^{-\bar{t}/\tau_J} C^{p-1}(\bar{t})C(t-\bar{t}) \mapsto \frac{1}{v} q^p, \quad (5.5)$$

where

$$q = \lim_{t \rightarrow \infty} (\lim_{\tau_J \rightarrow \infty} C(t)). \quad (5.6)$$

To investigate phase transitions we take our usual restriction to the parameters of the model: $v = T_J$, $n \equiv T/T_J$, $\mu = p\beta^2 J^2/2$. The long-time limit of (5.4) gives RS equation (4.5). Indeed, for determination of q we can take the two limits: $t = 0$ where $C(0) = 1$, $\partial_t C(t)|_{t=0} = -T$, and $t \mapsto \infty$ (in the sense of (5.6)) where $C = q$:

$$\beta r - 1 = \mu n q^p + \mu(1 - q^p) \quad (5.7)$$

$$\beta r q = \mu n q^p + \mu(q - q^p) + \mu q^{p-1}(1 - q). \quad (5.8)$$

As shown in Section 4, the non-zero solution of this equation correctly describes the low-temperature statics of the system for $n > 1$.

What is the meaning of a dynamical phase transition? Our starting point involves overdamped Langevin equations which are first order with respect of time. Thus, our dynamics is purely relaxational: $C(t)$ decays starting from $C(0) = 1$. In other words, we have: $\partial_t C(t) \leq 0$. One defines the dynamic phase transition point as the highest temperature, where this condition is broken for some $t > 0$ ². We obtain from (5.4)

$$\begin{aligned} (\partial_t + r)C(t) &= \mu T C(t) + (\beta r - \mu - 1)T \\ &+ \mu T C^{p-1}(t)(1 - C(t)) \\ &- \mu T \int_0^t d\bar{t} \partial_{\bar{t}} C(\bar{t}) [C^{p-1}(t-\bar{t}) - C^{p-1}(t)] \\ &+ \frac{pJ^2}{2\Gamma_J} \int_0^\infty d\bar{t} e^{-\bar{t}/\tau_J} C^{p-1}(\bar{t}) \\ &\times (C(t-\bar{t}) - C(\bar{t})). \end{aligned} \quad (5.9)$$

² The dynamical phase transition should not be mixed with transitions predicted by the dynamics in general. Within the common definition it is a kind of dynamical instability, whereas other “non-statical” scenarios are also possible [42].

In the last integral one can take $\exp(-\bar{t}/\tau_J) = 1$ according to equation (5.6); this integral converges, but can be neglected due to the large factor $1/\Gamma_J$. Now equation (5.9) reads

$$\begin{aligned} \partial_t C(t) &= T(1 - C(t)) \left[\beta r - \mu - \frac{1}{1 - C(t)} + \mu C^{p-1}(t) \right] \\ &- \mu T \int_0^t d\bar{t} \partial_{\bar{t}} C(\bar{t}) [C^{p-1}(t-\bar{t}) - C^{p-1}(t)]. \end{aligned} \quad (5.10)$$

The integral is negative for $\partial_t C(t) \leq 0$, and close to zero due to the factor $C^{p-1}(t-\bar{t}) - C^{p-1}(t)$ for large t . Thus, one gets

$$\beta r - \mu < \frac{1}{1 - C(t)} - \mu C^{p-1}(t). \quad (5.11)$$

The highest temperature, where this condition is broken, can be found with

$$\beta r - \mu = \frac{1}{1 - C(t)} - \mu C^{p-1}(t) \quad (5.12)$$

$$\frac{1}{[1 - C(t)]^2} - (p-1)\mu C^{p-2}(t) = 0. \quad (5.13)$$

On the other hand equation (5.11) should be consistent with equations (5.7, 5.8), and then with q determined by equation (4.5). One derives from equations (5.7, 5.8, 5.12, 5.13):

$$(n-1) \mu q^p = \frac{C(t)}{[1 - C(t)]^2} \left(\frac{p-2}{p-1} - C(t) \right). \quad (5.14)$$

As known from Section 4, for $T \geq T_{1,rs}$ only the paramagnetic solution $q = 0$ exists. Substituting this value to equations (5.14) one obtains a dynamical phase transition with a jump

$$q_d = \frac{p-2}{p-1}, \quad (5.15)$$

at the temperature

$$T_d = \sqrt{\frac{pJ^2}{2} \frac{(p-2)^{p-2}}{(p-1)^{p-1}}}. \quad (5.16)$$

This is exactly the dynamical transition obtained for the case $n = 0$ [14,35]. In our case it will exist only when $T_d \geq T_{1,rs}$, which is satisfied for $n < 1$, $p > 2$ and $n < 2$, $p = 2$ (in this last case T_d and $q_d = 0$ coincide with their statical values). For $T_d \leq T_{1,rs}$ the solution of equation (4.5) with $q > 0$ should be taken in equation (5.14), since it is the largest possible solution, and decaying from $C(0) = 1$ the correlation function $C(t)$ will stick there. That is the reason why the transition temperature $T_{2,rs}$ (Tab. 1, Eq. (4.21)) is not reflected in the dynamics. For $n \geq 1$ one has $1 \geq C(t) \geq q \geq q_{1,rs} \geq q_d = (p-2)/(p-1)$, namely equation (5.14) cannot be valid for $n > 1$, since the left-hand side is positive, and the right-hand side is

negative. Thus, for $n > 1$ and $q > 0$ there is no dynamical phase transition.

This conclusion is supported by the results obtained from the statics, since the dynamical phase transition can be reflected in the replica theory by the condition of marginality [21, 37, 47, 48]: The ‘‘replicon (ergodon)’’ eigenvalue of the Hessian is equal to zero at the transition point. This is an alternative scheme to fix the parameter m in the replica theory. For $n \leq 1$ the replicon eigenvalue is given by equation (4.12), and the condition of marginality

$$A_1 = -\mu(p-1)q^{p-2} + \frac{1}{(1-q)^2} = 0 \quad (5.17)$$

is nothing else, but equation (5.13) in the long-time limit. However, for $n > 1$ A_1 is no longer the most relevant (minimal) eigenvalue of Hessian, and condition (5.17) has no physical meaning there.

The standard interpretation of this phase transition is the following [17, 22, 14, 35–37]. The dynamics is investigated in the limits: the initial time $\rightarrow -\infty$ after the thermodynamic limit $N \rightarrow \infty$. In contrast, in the purely static investigation by means of the Gibbs distribution, the opposite order of limits is implicated. It was shown that the dynamical phase transition is induced by the highest metastable TAP states [36] which are separated by each other by means of infinite barriers (in the thermodynamic limit). Now it is clear that the existence and probably the structure of the highest metastable TAP states strongly depends on n .

5.2 Adiabatic dynamics: very large observation times

In this subsection we have concentrated on that range of n , where the dynamical phase transition is absent for all T . As the previous subsection shows, there is a spin-glass phase if the observation time is not very large: $t \ll \tau_J$. However, for $t \sim \tau_J$ the coupling constants will start to fluctuate and relax toward their own steady state. It is expected intuitively that at such observation times the spin-glass phase will disappear, and the resulting stationary state will be a paramagnet. Furthermore, we will see that there are some peculiar features of this process, which have universal character related to other glassy systems [14, 35].

In the domain $t \sim \tau_J \rightarrow \infty$ the main time-scale is τ_J ; thus, it is natural to take the following form for the correlation and response functions

$$C(t) = \mathcal{C}\left(\frac{t}{\tau_J}\right), \quad G(t) = \frac{1}{\tau_J} \mathcal{G}\left(\frac{t}{\tau_J}\right) \quad (5.18)$$

(the extra factor $1/\tau_J$ for \mathcal{G} appears on account of the correct dimension of that quantity.) Let us also make explicit a time-scale t_e , at which the correlation function stabilizes: $C(t \sim t_e) = q$, where q is defined by equation (4.5). By its definition t_e is much higher than the relaxation time of the spins, but much lower than the relaxation time of the coupling constants ($\tau_J \gg t_e$). Later one will see that a more precise value of t_e is not needed.

The cases $t \sim t_e$ and $t < t_e$ have been studied in the previous subsection, and now we are going to consider the cases $t \sim \tau_J$, $t \gg \tau_J$. Since the technical part of our derivation is quite similar to those given in [14, 35], we shall mention only the main points. The corresponding equations for \mathcal{C} , \mathcal{G} must be constructed from (5.1, 5.2) in accordance of a simple physical picture occurring from the large separation of the local relaxation times. Namely, if the temporal argument of $C(\bar{t})$, $G(\bar{t})$ is less or equal t_e , then the case of not very large observation times holds with all its consequences. For instance, the usual FDT can be used. At the same time the kernel $\exp(-\bar{t}/\tau_J)$ can be put equal to unity, as we have done in the previous subsection. If the corresponding temporal argument has the same order as t_e we put $C(t \sim t_e) = q$, and if it has the same order as τ_J we use (5.18) accompanying with an evident consistency condition $\mathcal{C}(0) = q$.

Our transformations come to dividing the domains of integration in equation (5.1) into two parts: from 0 to t_e , and from t_e to ∞ . The corresponding integrals in equation (5.2) are divided into three parts because more accuracy is needed: from 0 to t_e , from t_e to $t - t_e$, and from $t - t_e$ to ∞ . Now every part is treated as described above. We omit all factors which have relatively small order when $\tau_J \rightarrow \infty$. In particular, for equation (5.1, 5.2) these factors are of order $\mathcal{O}(1/\tau_J)$ ($\mathcal{O}(1/\tau_J^2)$).

Finally, we get the following equations (where $s = t/\tau_J$)

$$\begin{aligned} r\mathcal{C}(s) &= \frac{pJ^2}{2v} \int_0^\infty d\bar{s} e^{-\bar{s}} \mathcal{C}^{p-1}(\bar{s}) \mathcal{C}(s - \bar{s}) \\ &+ \frac{p(p-1)T_J J^2}{2v} \int_0^\infty d\bar{s} e^{-\bar{s}} \mathcal{C}^{p-2}(\bar{s}) \mathcal{G}(\bar{s}) \mathcal{C}(s - \bar{s}) \\ &+ \frac{pT_J J^2}{2v} \int_0^\infty d\bar{s} e^{-(s+\bar{s})} \mathcal{C}^{p-1}(s + \bar{s}) \mathcal{G}(\bar{s}) \\ &+ \frac{pT_J J^2}{2v} \beta e^{-s} \mathcal{C}^{p-1}(s) (1 - q) \\ &+ \frac{pT_J J^2}{2v} \beta \mathcal{C}(s) (1 - q^{p-1}), \end{aligned} \quad (5.19)$$

$$\begin{aligned} r\mathcal{G}(s) &= \frac{pJ^2}{2v} \int_0^s d\bar{s} e^{-\bar{s}} \mathcal{C}^{p-1}(\bar{s}) \mathcal{G}(s - \bar{s}) \\ &+ \frac{p(p-1)T_J J^2}{2v} \int_0^s d\bar{s} e^{-\bar{s}} \mathcal{C}^{p-2}(\bar{s}) \mathcal{G}(s - \bar{s}) \mathcal{G}(\bar{s}) \\ &+ \frac{pT_J J^2}{2v} \beta \mathcal{G}(s) (1 - q^{p-1}) \\ &+ \frac{p(p-1)T_J J^2}{2v} \beta e^{-s} \mathcal{C}^{p-2}(s) \mathcal{G}(s) (1 - q) \\ &+ \frac{pJ^2}{2v} \beta e^{-s} \mathcal{C}^{p-1}(s) (1 - q). \end{aligned} \quad (5.20)$$

The usual formulation of FDT does not hold in this case. Indeed, if the coupling constants fluctuate, then the heat current between the thermal baths cannot be neglected. In other words, the detailed balance condition is violated, because there is a stationary current between the two heat

baths, which changes its sign under time-reversal. Thus, we have a steady non-Gibbsian state [19]. In general, such a state does not possess any simple, or even closed, relation between the correlation and response functions [3]. However, such a relation is possible in some particular cases. In our case a generalized FDT exists, which is, however, non-universal in contrast to the usual one. Nevertheless, it does not depend on secondary details of the model. As such it belongs to the thermodynamical picture of the glassy state [15,41]. Let us consider the generalized FDT in the form

$$\partial_s \mathcal{C}(s) = \tilde{T}(-\mathcal{G}(s) + \mathcal{G}(-s)) \quad (5.21)$$

with some unknown coefficient \tilde{T} . After some calculations using (5.19), (5.20) we get

$$\tilde{T} = T_J. \quad (5.22)$$

It is customary to call (5.21) ‘‘Fluctuation-Dissipation Relation’’ (FDR) or just the ‘‘modified FDT’’. We have obtained that the coefficient of FDR (it is T_J in the case of Eq. (5.22), and T in the case of (5.3)) depends crucially on the observation time: The correspondence between the time-scales and the temperatures is reflected in FDR. Such phenomena were predicted recently for systems with ‘‘slow’’ dynamics [14,15,41]. In particular, this is a possible fundament to generalize the notion of temperature. With help of the FDR we obtain the following formula for $\mathcal{G}(0)$:

$$\mathcal{G}(0) = \frac{\beta q(1-q)}{q - (p-2)\beta T_J(1-q)}. \quad (5.23)$$

Now the positivity of this quantity requires condition (4.15), which is necessary for the validity of the RS assumption. Since $\mathcal{G}(s)$ decreases from $\mathcal{G}(0)$ to zero, the consistency condition is nontrivial only for $q > 0$.

Let us now discuss the limit $t \gg \tau_J$. We rewrite equation (5.19) taking into account (5.21, 5.22), and the concrete value of r :

$$(1-q)(q^{p-2} - e^{-s}\mathcal{C}^{p-2}(s))\mathcal{C}(s) = T \int_0^s d\bar{s} e^{-\bar{s}} \mathcal{C}^{p-1}(\bar{s})\mathcal{G}(s-\bar{s}). \quad (5.24)$$

The static limit of this equation correspond to large s :

$$\lim_{\tau_J \rightarrow \infty} (\lim_{t \rightarrow \infty} \mathcal{C}(t)) = \lim_{s \rightarrow \infty} \mathcal{C}(s) = 0, \quad (5.25)$$

because in the relevant part of integral (5.24) the function \mathcal{G} is zero: When $t/\tau_J \rightarrow \infty$ every correlation will vanish at large times, and the system goes to the paramagnetic phase. It is just that statement which was predicted above starting with heuristic arguments.

There is no spin-glass phase if the spins and the coupling constants have nearly equal characteristic times. It is quite obvious in the light of the present discussion.

6 Summary

This paper is devoted to a glassy system coupled to two heat baths. In Section 3 we use the adiabatic assumption to construct a generalized thermodynamic. Its basic relation (3.17) involves the entropies of the spins and the coupling constants, which in the present approach have the independent and well-defined physical meaning. The usual local-equilibrium thermodynamics [3,5] is recovered with weakly-interacting subsystems, or close temperatures T, T_J . The developed theory has a general character, and does not depend on concrete details of the considered systems. After this, in Section 4 this theory is applied to the mean-field p -spin-interaction spherical model, extended to have correlated random bonds, expressed by a finite temperature T_J . In the limit $T_J \rightarrow \infty$, so $n = T/T_J \rightarrow 0$, the usual spin-glass model with totally uncorrelated bonds is recovered. As noted recently [42], this type of correlations can make radical changes in the phase structure. In this context, the p -spin model is the convenient laboratory for investigating phase transitions, since it belongs to different universality classes for $p > 2$ and $p = 2$. Indeed, if n is large enough there are only first-order phase transitions with positive latent heat (see Figs. 7, 6). This is in the contrast to the first-order transitions without latent heat ($p > 2$) or the true second-order transitions ($p = 2$), which are more typical for spin glasses and glasses, and realized in the remaining parts of the phase diagram. The 1RSB (replica symmetry breaking) spin-glass phase can exist as a truly stable phase only for $m < n < 1$ (see Eqs. (4.30, 4.31)). Replica symmetry is always restored for sufficiently low temperatures and $n > 0$. Notice the differences compared to the SK model with infinite-order RSB, where a finite n mainly modifies the existing spin-glass phase, introducing the lower plateau for the order parameter $q(x)$ [29,28]. This plateau grows with n , and RSB disappears at some critical value n_c ($n_c < 1$). Nearly the same behavior is introduced by an external magnetic field. In our case such a phase does not exist at all, and the 1RSB phase with vanishing lower plateau exists even for $n > 1$, but only as a metastable phase. These distinctions are connected with different structures of the phase space.

For all $p \geq 2$ the first-order phase transitions are related with an interesting effect: In spite of the fact that the jump of the mean energy u at the transition point is negative (because the latent heat is positive) the jump of the total entropy is positive. This uncommon property is possible only due to our generalized thermodynamical relation (3.17), combined with the fact that the corresponding jumps of s_σ and s_J have opposite signs: $\Delta s_\sigma < 0$, $\Delta s_J > 0$, but the sum $s = s_J + s_\sigma$ has a positive jump (see Fig. 5). The situation is slightly different for the first-order type phase transition without latent heat. In Section 4.2.1 we have seen that the total entropy increases continuously in the course of the phase transition from the paramagnet to 1RSB spin-glass (see Fig. 2). In this respect an interesting analogy exists with the process of coarse(fine)-graining (see the discussion after Eq. (3.13)). Further developments of these analogies will be quite interesting.

There is a close relation between free energies and saddle-point equations of the finite n RS case and the $q_0 = 0$ case of the 1RSB equations. The RS solution with nonzero n corresponds to the 1RSB solution with $q_0 = 0$ [25]. This mechanism is responsible for the transition between the 1RSB and RS spin-glass phases. The transition is second-order with respect to free energy and its derivatives, but is connected with a jump of q_0 from zero to q_1 that ensures replica-symmetric behavior (see Figs. 3, 4). This analogy exists in dynamics also: in our time-translation invariant two-temperature dynamics (the invariance is valid for temperatures higher than the dynamical transition point) there is the phenomena similar to the generalized FDT in the $n = 0$ non-equilibrium dynamics [14]. This effect reflects intrinsic connections between systems where the complex structure of the phase space is self-generated and there are different time-scales for the global relaxation (ergodicity breaking), and systems where different components have different temperatures and relaxation-scales.

For $n < 1$ and $p > 2$ there is the dynamical phase transition occurring at the higher temperature compared to the static transitions. This known effect is due to the existence of the whole set of metastable states with free energies greater than the free energy of the pure states predicted by statics. It is connected with the absence of activated processes on the timescales considered in the dynamics [18, 21, 16], which enters due to mean-field (infinite dimension) character of the model [45]. In the static consideration we observe times $\rightarrow \infty$ before the thermodynamic limit $N \rightarrow \infty$, but if dynamics is investigated by means of generating functional [49] the first limit is taken after the second, which eliminates activated process that need time-scales exponential in N . In the corresponding finite-dimensional systems a smoothening of this effect is expected, where instead a sharp phase transition a near-critical domain of temperatures will take place. Notice also that for $n = 0$ and $p = 2$ the RS spin-glass phase is only marginally stable. A non-zero n stabilizes the corresponding fluctuations.

The predictions of the adiabatic statics and dynamics can be compared only for the relatively short observation times. The spin-glass phase appears at times $\ll \tau_J$ (the characteristic time of the coupling constants), but disappears for the observation times $\gg \tau_J$. In this limit of long observations the coupling constants cannot be viewed as frozen. This is the non-equilibrium steady state without any spin-glass ordering (only a critical slowing down of the spin-spin correlation function occurs when $T_J \rightarrow 0$). In this respect it is similar to weak ergodicity breaking occurring in the non-equilibrium dynamics [14]. A generalized fluctuation-dissipation relation has been proven, which contains the temperature of the couplings instead of the temperature of the spins. This relation is also closely connected to the non-equilibrium generalization of the FDT [14, 15], where, in particular, the asymptotic long-time non-equilibrium state of the $n = 0$ p -spin spherical model is considered.

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Appendix A

In this appendix we consider how to get the usual Langevin equations of the spin-glass model with *a priori* random freezed coupling constants from (2.2, 2.3).

Equation (2.3) can be solved exactly with respect of $J_{i_1 \dots i_p}$:

$$J_{i_1 \dots i_p}(t) = J_{i_1 \dots i_p}^{(0)} e^{-(t-t_0)/\tau_J} + A_{i_1 \dots i_p}(t) + B_{i_1 \dots i_p}(t) \quad (\text{A.1})$$

$$A_{i_1 \dots i_p}(t) = \frac{1}{\Gamma_J} \int_{t_0}^t d\bar{t} e^{-(t-\bar{t})/\tau_J} \eta_{i_1 \dots i_p}(\bar{t}), \quad (\text{A.2})$$

$$B_{i_1 \dots i_p}(t) = \frac{1}{\Gamma_J} \int_{t_0}^t d\bar{t} e^{-(t-\bar{t})/\tau_J} J_N \sigma_{i_1}(\bar{t}) \dots \sigma_{i_p}(\bar{t}), \quad (\text{A.3})$$

where $J_{i_1 \dots i_p}^{(0)}$ are the initial conditions at the moment $t = t_0$, $\tau_J = \Gamma_J J_N^2 / v$, and $J_N^2 = p! J^2 / 2N^{p-1}$. The initial factors can be neglected if $t_0 \mapsto -\infty$, $|t_0| \gg \tau_J$. Further we have

$$\langle A_{i_1 \dots i_p}(t) A_{j_1 \dots j_p}(t') \rangle = \frac{T_J J_N^2}{v} \delta_{i_1 \dots i_p, j_1 \dots j_p} e^{-|t-t'|/\tau_J}. \quad (\text{A.4})$$

Thus if $\tau_J \gg t - t'$, then $A_{i_1 \dots i_p}$ is a quenched Gaussian noise, and if $v \sim T_J$ and $T_J \mapsto \infty$ then $B_{i_1 \dots i_p}$ can be neglected with respect of $A_{i_1 \dots i_p}$. So in these limits $J_{i_1 \dots i_p}$ is a quenched Gaussian noise. Each coupling constant is independent of the other ones and the spins.

Appendix B

In this appendix we discuss the derivation of (2.4) from the initial Langevin equations. We investigate these equations by the dynamical generating functional method [49, 21].

$$\begin{aligned} 1 &= Z_{\text{dyn}} \\ &= \int \prod_i D\eta_i \prod_{[i_1 \dots i_p]} D\eta_{i_1 \dots i_p} \exp\left(-\frac{1}{4\Gamma T}\right. \\ &\quad \left. \times \sum_i \int dt \eta_i^2(t) - \frac{1}{4\Gamma_J T_J} \sum_{[i_1 \dots i_p]} \int dt \eta_{i_1 \dots i_p}^2(t)\right), \end{aligned} \quad (\text{B.1})$$

where $[i_1 \dots i_p] = 1 \leq i_1 < \dots < i_p \leq N$, and normalization factors are included in $D\eta_i$, $D\eta_{i_1 \dots i_p}$. By means of Z_{dyn} we can compute the noise average of any quantity

$A(\{\sigma\}, \{J\})$:

$$\begin{aligned} \langle A \rangle &= \int D\sigma D J \left[\frac{\delta \eta}{\delta \sigma} \right] \left[\frac{\delta \eta}{\delta J} \right] A(\{\sigma\}, \{J\}) \\ &\times \exp \left[-\frac{1}{4\Gamma T} \sum_i \int dt \left(\Gamma \partial_t \sigma_i + r \sigma_i + \frac{\partial \mathcal{H}}{\partial \sigma_i} \right)^2 \right. \\ &\left. - \frac{1}{4\bar{\Gamma}_J T_J} \sum_{[i_1 \dots i_p]} \int dt \left(\bar{\Gamma}_J \partial_t J_{i_1 \dots i_p} + \frac{\partial \mathcal{H}}{\partial J_{i_1 \dots i_p}} \right)^2 \right] \end{aligned} \quad (\text{B.2})$$

where $[\delta \eta / \delta \sigma]$, $[\delta \eta / \delta J]$ are the corresponding functional Jacobians. After a simple transformation we have

$$\begin{aligned} Z_{\text{dyn}} &= \int D\sigma D J D \hat{\sigma} D \hat{J} \exp \left(-\Gamma T \sum_i \int dt \hat{\sigma}_i^2 \right. \\ &+ i \sum_i \int dt \hat{\sigma}_i(t) \left(\Gamma \partial_t \sigma_i + r \sigma_i + \frac{\partial \mathcal{H}}{\partial \sigma_i} \right) \Big) \\ &\times \exp \left(\sum_{[i_1 \dots i_p]} \left[-\bar{\Gamma}_J T_J \int dt \hat{J}_{i_1 \dots i_p}^2(t) \right. \right. \\ &+ i \int dt \hat{J}_{i_1 \dots i_p}(t) \left(\bar{\Gamma}_J \partial_t J_{i_1 \dots i_p} + \frac{\partial \mathcal{H}}{\partial J_{i_1 \dots i_p}} \right) \Big] \Big) \\ &\times \exp(V_{\eta\sigma} + V_{\eta J}). \end{aligned} \quad (\text{B.3})$$

Here the last exponent is the contribution of the functional Jacobians; we shall not write this expression explicitly because it is not relevant in the mean-field approximation.

We want to derive equations for spin dependent functions, therefore in equation (B.1) we can take Gaussian integrations by $\{\sigma\}$, $\{\hat{J}\}$. The result is

$$\begin{aligned} Z_{\text{dyn}} &= \int D\sigma D \hat{\sigma} \exp \left[-\Gamma T \sum_i \int dt \hat{\sigma}_i(t) \right. \\ &+ i \sum_i \int dt \hat{\sigma}_i(t) (\Gamma \partial_t + r) \sigma_t \\ &- \frac{i J_N^2}{\Gamma_J} \sum_{[i_1 \dots i_p]} \int dt dt' \phi(t-t') a_{i_1 \dots i_p}(\sigma, \hat{\sigma}, t') b_{i_1 \dots i_p}(\sigma, t) \\ &- \frac{T_J J_N^2}{2v} \sum_{[i_1 \dots i_p]} \int dt dt' k(t-t') a_{i_1 \dots i_p}(\sigma, \hat{\sigma}, t') a_{i_1 \dots i_p}(\sigma, \hat{\sigma}, t) \Big] \\ &\times \exp(V_{\eta\sigma} + V_{\eta J}), \end{aligned} \quad (\text{B.4})$$

where

$$\begin{aligned} a_{i_1 \dots i_p}(\sigma, \hat{\sigma}, t) &= \sum_{s=1}^p \sigma_{i_1}(t) \dots \sigma_{i_{s-1}}(t) \hat{\sigma}_{i_s}(t) \sigma_{i_{s+1}}(t) \dots \sigma_{i_p}(t), \\ b_{i_1 \dots i_p}(\sigma, t) &= \sigma_{i_1}(t) \dots \sigma_{i_p}(t), \end{aligned} \quad (\text{B.5})$$

$$\begin{aligned} \phi(t-t') &= \theta(t-t') e^{-(t-t')/\tau_J}, \\ k(t-t') &= e^{-|t-t'|/\tau_J}, \quad \tau_J = \Gamma_J/v. \end{aligned} \quad (\text{B.6})$$

By the standard mean-field procedure [21] we introduce the following order parameters:

$$\begin{aligned} Q_1(t, t') &= \langle i \hat{\sigma}(t) i \hat{\sigma}(t') \rangle, \\ Q_2(t, t') &= \langle \sigma(t) \sigma(t') \rangle, \\ Q_3(t, t') &= \langle \sigma(t) i \hat{\sigma}(t') \rangle, \\ Q_4(t, t') &= \langle \sigma(t') i \hat{\sigma}(t) \rangle, \end{aligned} \quad (\text{B.7})$$

and the corresponding Lagrange factors $\lambda_s(t, t')$, $s = 1, \dots, 4$. In this scheme Q_2 is the correlation function, Q_3, Q_4 are the susceptibilities; Q_1 (“field-field” correlation function) should be taken zero by reasons of causality [21]. Now we have

$$\begin{aligned} Z_{\text{dyn}} &= \int \prod_{s=1}^4 \frac{D\lambda_s D Q_s}{2\pi i} \exp \left(N \Omega(\lambda_s, Q_s) \right. \\ &+ N \int D\sigma D \hat{\sigma} e^{\mathcal{L}(\sigma, \hat{\sigma})} \Big) \\ \Omega(\lambda_s, Q_s) &= - \int dt dt' \sum_{s=1}^4 \lambda_s(t, t') Q_s(t, t') \\ &+ \frac{p J^2}{2\Gamma_J} \int dt dt' \phi(t-t') Q_3(t, t') Q_2^{p-1}(t, t') \\ &+ \frac{p T_J J^2}{4v} \int dt dt' \phi(t-t') [Q_1(t, t') Q_2^{p-1}(t, t') \\ &+ (p-1) Q_3(t, t') Q_4(t, t') Q_2^{p-2}(t, t')], \end{aligned} \quad (\text{B.8})$$

where

$$\begin{aligned} \mathcal{L}(\sigma, \hat{\sigma}) &= -\Gamma T \sum_i \int dt \hat{\sigma}_i(t) - i \sum_i \int dt \hat{\sigma}_i(t) (\Gamma \partial_t + r) \sigma_t \\ &+ \lambda_1(t, t') i \hat{\sigma}(t) i \hat{\sigma}(t') + \lambda_2(t, t') \sigma(t) \sigma(t') \\ &+ \lambda_3(t, t') i \sigma(t) \hat{\sigma}(t') + \lambda_4(t, t') i \hat{\sigma}(t) \sigma(t'). \end{aligned} \quad (\text{B.9})$$

By variational methods we obtain

$$\begin{aligned} \lambda_1(t, t') &= \frac{p T_J J^2}{4v} k(t-t') Q_2^{p-1}(t, t') \\ \lambda_2(t, t') &= \frac{p(p-1) J^2}{2\Gamma_J} \phi(t-t') Q_3(t, t') Q_2^{p-2}(t, t') \\ &+ \frac{p T_J J^2}{4v} k(t-t') ((p-1) Q_1(t, t') Q_2^{p-2}(t, t') \\ &+ (p-1)(p-2) Q_3(t, t') Q_4(t, t') Q_2^{p-3}(t, t')) \\ \lambda_3(t, t') &= \frac{p J^2}{2\Gamma_J} \phi(t-t') Q_2^{p-1}(t, t') \\ &+ (p-1) \frac{p T_J J^2}{4v} k(t-t') Q_4(t, t') Q_2^{p-2}(t, t') \\ \lambda_4(t, t') &= (p-1) \frac{p T_J J^2}{4v} k(t-t') Q_3(t, t') Q_2^{p-2}(t, t'), \end{aligned} \quad (\text{B.10})$$

λ_2 can be adsorbed in the Jacobians. These results should be substituted to (B.9): the effective dynamics of a spin is determined by spins motion at the environment of the spin, and by motion of the coupling constants. After this lengthy calculation we arrive at (2.4).

References

1. M. Mezard, G. Parisi, M.A. Virasoro, *Spin-Glass Theory and Beyond* (World Scientific, Singapore, 1986).
2. K. Binder, A.P. Young, *Rev. Mod. Phys.* **58**, 801 (1986).
3. R.L. Stratonovich, *Nonlinear Nonequilibrium Thermodynamics I* (Springer, 1994).
4. L.D. Landau, E.M. Lifshitz, *Statistical Physics, Part 1* (Pergamon Press, 1980).
5. S.R. de Groot, P. Mazur, *Nonequilibrium Thermodynamics* (North-Holland, Amsterdam, 1962).
6. P.G. Bergmann, J.L. Lebowitz, *Phys. Rev.* **99**, 578 (1955); H. Spohn, J.L. Lebowitz, in *Advances in Chemical Physics*, edited by S.A. Rice (John Wiley and Sons, 1978).
7. J. Meixner, in *A Critical Review of Thermodynamics*, edited by E.B. Stuart (Oxford University Press, 1970).
8. M. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1982).
9. E. Nagaev, *Magnets with Complicate Exchange Interaction* (Nauka, Moscow, 1988).
10. A.S. Andreenko, S.A. Nikitin, *Uspekhi Fizicheskikh Nauk* **167**, 605 (1997).
11. V.L. Pokrovskii, A.Z. Patashinskii, *Fluctuation Theory of Phase Transitions* (Nauka, Moscow, 1982).
12. R.O. Davies, G.O. Jones, *Adv. Phys.* **2**, 370 (1953).
13. J. Jäckle, *Philos. Mag. B* **44**, 533 (1981).
14. L.F. Cugliandolo, J. Kurchan *Phys. Rev. Lett.* **71**, 173 (1993); *Philos. Mag. B* **71**, 50 (1995).
15. L. F. Cugliandolo, J. Kurchan, L. Peliti, *Phys. Rev. E* **55**, 3898 (1997); L.F. Cugliandolo, J. Kurchan *cond-mat/9807226*.
16. Th.M. Nieuwenhuizen, *J. Phys. A. Lett.* **31**, L201 (1998).
17. J.-P. Bouchaud, L.F. Cugliandolo, J. Kurchan, M. Mezard, in *Spin-Glasses and Random Fields*, edited by A.P. Young, (World Scientific, 1998).
18. W. Götze, L. Sjogren, *Rep. Prog. Phys.* **55**, 241 (1992); W. Götze, *J. Phys. Cond. Mat.* **11**, 1 (1999).
19. For a review and further references about phase transitions in nonequilibrium steady states, see: B. Schmittmann, R.K.P. Zia, in *Phase Transitions and Critical Phenomena*, edited by C. Domb, J.L. Lebowitz (Academic Press, New York, 1996).
20. M. Roger, M. Debieu, J.H. Hetherington, *Rev. Mod. Phys.* **55**, 1 (1983).
21. T.R. Kirkpatrick, P.G. Wolynes, *Phys. Rev. A* **34**, 1045 (1986); T.R. Kirkpatrick, D. Thirumalai, *Phys. Rev. B* **36**, 5388 (1987); T.R. Kirkpatrick, D. Thirumalai, P.G. Wolynes, *Phys. Rev. A* **40**, 2047 (1989).
22. H. Horner, *Z. Phys. B* **57**, 29 (1984); *ibid* **57**, 39 (1984).
23. R. Landauer, J.W.F. Woo, *Phys. Rev. A* **6**, 2205 (1972).
24. A.C.C. Coolen, R.W. Penney, D. Sherrington, *J. Phys. A* **26**, 3681 (1993).
25. R.W. Penney, D. Sherrington, *J. Phys. A* **27**, 4027 (1994).
26. G. Jongen, D. Bolle, A.C.C. Coolen, *J. Phys. A* **31**, L737-L742 (1998).
27. N. Caticha, *J. Phys. A* **27**, 5501 (1994).
28. Vik. Dotsenko, M. Mezard, S. Franz, *J. Phys. A* **27**, 2351 (1994).
29. I. Kondor, *J. Phys. A* **16**, L127 (1983).
30. J.M. Kosterlitz, D.J. Thouless, R.C. Jones, *Phys. Rev. Lett.* **36**, 1217 (1976).
31. E. Gardner, *Nucl. Phys. (B)[FS14]* **257**, 747 (1985).
32. A. Crisanti, H.J. Sommers, *Z. Phys. B* **87**, 341 (1992).
33. J. Kurchan, G. Parisi, M.A. Virasoro, *J. Phys. I France* **3**, 1819 (1993).
34. H. Sompolinsky, *Phys. Rev. Lett.* **47**, 935 (1981); *Philos. Mag. B* **50**, 285 (1984).
35. A. Crisanti, H. Horner, H.J. Sommers, *Z. Phys. B* **92**, 257 (1993).
36. A. Crisanti, H.J. Sommers, *J. Phys. I France* **5**, 805 (1995).
37. Th.M. Nieuwenhuizen, *Phys. Rev. Lett.* **74**, 3463 (1995).
38. Th.M. Nieuwenhuizen, *Complexity as driving force for dynamical glassy transition; cond-mat/9504059*.
39. Th.M. Nieuwenhuizen, *Phys. Rev. Lett.* **79**, 1317 (1997).
40. Th.M. Nieuwenhuizen, *Phys. Rev. Lett.* **80**, 5581 (1998).
41. Th.M. Nieuwenhuizen, *Phys. Rev. E* **61**, 267 (2000).
42. A.E. Allahverdyan, D.B. Saakian, *Phys. Rev. E* **58**, 5201 (1998).
43. A.E. Allahverdyan, Th.M. Nieuwenhuizen, *cond-mat/9907143*, to be published in *Phys. Rev. E* (July 2000).
44. D.J. Gross, I. Kanter, H. Sompolinsky, *Phys. Rev. Lett.* **55**, 304 (1985).
45. G. Parisi, *Slow Dynamics of Glassy Systems., cond-mat/9705312*.
46. A. Houghton, S. Jain, A.P. Young, *Phys. Rev. B* **28**, 2630 (1983).
47. H. Horner, *Z. Phys. B*, **66**, 175, (1987).
48. J. Kurchan, L. Laloux, *J. Phys. A* **29**, 1929 (1996).
49. H.K. Janssen, *Z. Phys. B* **26**, 187 (1977); C. De Dominicis, L. Peliti, *Phys. Rev. Lett.* **38**, 505 (1977); *Phys. Rev. B* **18**, 353 (1978).
50. C.H. Bennett, *Sci. Am.* **257**, 108 (1987); S. Lloyd, *Phys. Rev. A* **39**, 5378 (1989).
51. W. Ebeling, *Physica A* **182**, 108 (1992).
52. P. Shukla, S. Singh, *J. Phys. C* **14**, L81 (1981); S. Ciuchi, F. de Pasquale, *Nucl. Phys. B* **300** [FS 22], 31 (1988); Th.M. Nieuwenhuizen, *Phys. Rev. B* **31**, 7487 (1985); L.F. Cugliandolo, D.S. Dean, *J. Phys. A* **28**, 4213 (1995).